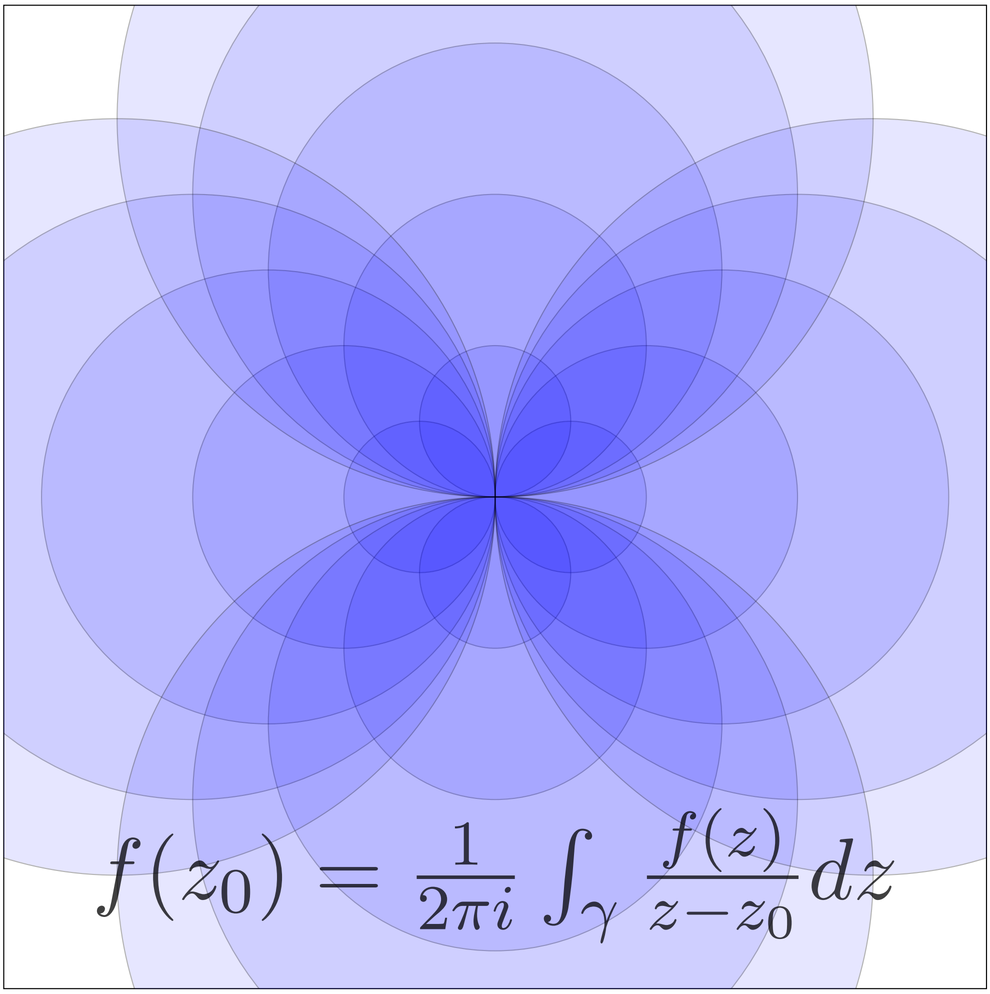
Lecture Notes on Complex Analysis



Markus Upmeier

Dept. of Mathematics, University of Aberdeen

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Contents

Chapter I

Complex numbers and functions

1 Geometry of the complex plane

This section is a brief reminder of Sections 3 and 4 of MA1006 Algebra.

Definition 1.1. The **complex numbers** $\mathbb{C}$ are the set of all pairs $z = (x, y) \in \mathbb{R}^2$ of real numbers with the addition

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1.1}$$

and the multiplication

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1), \tag{1.2}$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. We call $x = \mathrm{Re}(z)$ the **real part** and $y = \mathrm{Im}(z)$ the **imaginary part** of z.

From Section 3.8 in MA1006 Algebra we recall:

Proposition 1.2. *The complex numbers are a field.*

Notation 1.3. We view the real numbers as a subset of $\mathbb{C}$ by identifying $x \in \mathbb{R}$ with $(x, 0) \in \mathbb{C}$. The **imaginary unit** is $i = (0, 1)$. With these conventions, a calculation using (1.2) shows that

$$z = x + iy. \tag{1.3}$$

Using this notation, we can manipulate complex numbers in the same way as real numbers, keeping in mind the identity

$$i^2 = -1. \tag{1.4}$$

Of course, $\mathbb{C}$ is a one-dimensional vector space over itself. Restricting the scalar multiplication to $\mathbb{R}$ makes $\mathbb{C}$ into a vector space over $\mathbb{R}$, isomorphic to $\mathbb{R}^2$, of dimension two with standard basis $1, i \in \mathbb{C}$.

Proposition 1.4. (a) *In the standard basis, every* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{R})$ *corresponds uniquely to an* $\mathbb{R}$*-linear map* $T_A \colon \mathbb{C} \to \mathbb{C}$, *namely*

$$T_A(x + iy) = (ax + by) + i(cx + dy). \tag{1.5}$$

(b) T_A *is* $\mathbb{C}$*-linear* $\iff$ $a = d$ *and* $b = -c$. *In this case,*

$$T_A(z) = \alpha \cdot z, \qquad \alpha = a + ic. \tag{1.6}$$

Proof. (a) Recall from linear algebra that in a basis of a two-dimensional vector space linear transforms are represented by matrices $A \in \mathrm{M}_{2\times 2}(\mathbb{R})$. In the same way, $\mathbb{C}$-linear maps correspond to $\alpha \in \mathrm{M}_{1\times 1}(\mathbb{C})$ as in (1.6). An $\mathbb{R}$-linear map T_A is $\mathbb{C}$-linear $\iff$ $T_A(i) = T_A(i1) = iT_A(1)$ $\iff$

$$\begin{pmatrix} b \\ d \end{pmatrix} = T_A(i) = iT_A(1) = i\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -c \\ a \end{pmatrix}.$$

In this case, $T_A(x + iy) = (ax - cy) + i(cx + ay) = \alpha(x + iy)$. $\square$

Definition 1.5. The **conjugate** of $z \in \mathbb{C}$ is the complex number

$$\overline{z} = (x, -y), \tag{1.7}$$

and the **modulus** (also called **absolute value**) is

$$|z| = \sqrt{x^2 + y^2} \geqslant 0. \tag{1.8}$$

Proposition 1.6. *The following formulas hold for* $z, w \in \mathbb{C}$:

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad \overline{z + w} = \overline{z} + \overline{w} \tag{1.9}$$

$$\overline{\overline{z}} = z \qquad \overline{i} = -i, \ \ \overline{1} = 1 \tag{1.10}$$

$$z \cdot \overline{z} = |z|^2 \qquad |z \cdot w| = |z| \cdot |w| \tag{1.11}$$

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i} \tag{1.12}$$

$$z^{-1} = \frac{\overline{z}}{|z|^2} \quad \textit{if } z \neq 0 \tag{1.13}$$

Proposition 1.7. *The following inequalities hold for* $z, w \in \mathbb{C}$:

$$|z + w| \leqslant |z| + |w| \qquad |z - w| \geqslant \big||z| - |w|\big| \tag{1.14}$$

$$|\mathrm{Re}(z)| \leqslant |z| \qquad |\mathrm{Im}(z)| \leqslant |z| \tag{1.15}$$

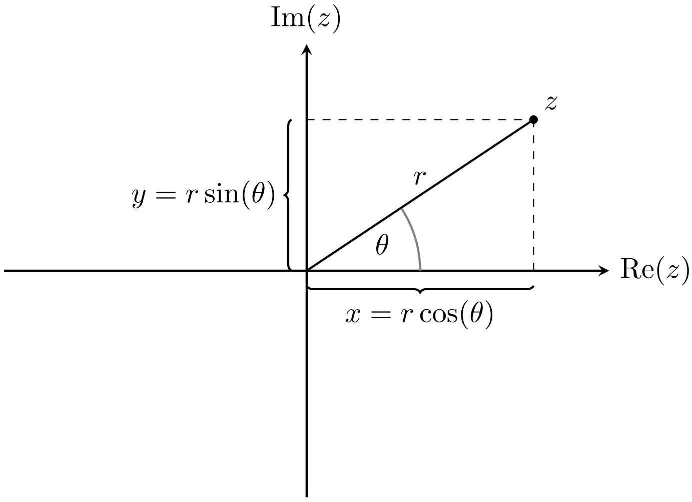

Figure 1.1: Polar coordinates

Thinking of complex numbers as points in the plane, we can use **polar coordinates** to represent them (see Figure 1.1).

Proposition 1.8. *For every non-zero complex number $z = (x, y)$ there is an* ***argument*** *$\theta \in \mathbb{R}$ and a* ***radius*** *$r = |z| > 0$ such that*

$$x = r\cos(\theta), \qquad\qquad y = r\sin(\theta). \tag{1.16}$$

This representation is unique up to replacing θ by $\theta + 2\pi k$ for any $k \in \mathbb{Z}$.

Proof. (may be omitted) Since $x^2 + y^2 = r^2(\cos(\theta)^2 + \sin(\theta)^2) = r^2$, the radius must be $r = |z|$. Define the complex number $w = r^{-1}z$ and write $w = u + iv$ for its real and imaginary parts.

We will prove the existence of $\theta \in \mathbb{R}$ with $u = \cos(\theta)$, $v = \sin(\theta)$. This also proves the existence of a representation (1.16), by multiplying by r. Since $u^2 + v^2 = |w|^2 = r^{-2}|z|^2 = 1$, we know $|u| \leqslant 1$, $|v| \leqslant 1$. Recall that $\cos\colon [0, \pi] \to [-1, 1]$ and $\sin\colon [-\pi/2, \pi/2] \to [-1, 1]$ are bijections. Hence

$$\begin{aligned} u &= \cos(\alpha) && \text{for some } \alpha \in [0, \pi], \\ v &= \sin(\beta) && \text{for some } \beta \in [-\pi/2, \pi/2]. \end{aligned}$$

As $\sin(\beta)^2 = v^2 = 1 - u^2 = 1 - \cos(\alpha)^2 = \sin(\alpha)^2$, we have $\sin(\alpha) = \pm\sin(\beta) = \sin(\pm\beta)$. To produce the correct θ, we distinguish two cases.

Case 1 $\alpha \in [0, \pi/2]$. Then $\alpha = \pm\beta$ by the injectivity of the sine function on the interval $[-\pi/2, \pi/2]$. Setting $\theta = \pm\alpha = \beta$, we find that $u = \cos(\theta)$ and $v = \sin(\theta)$, as required.

Case 2 $\alpha \in [\pi/2, \pi]$. Then $\pi - \alpha, \beta \in [-\pi/2, \pi/2]$ and $\sin(\pi - \alpha) = \sin(\alpha) = \pm\sin(\beta)$, so $\pi - \alpha = \pm\beta$ by injectivity. Setting $\theta = \pm\alpha = \pm\pi - \beta$, we find $u = \cos(\theta), v = \sin(\theta)$, using trigonometric identities.

This completes the existence part of the proof. For uniqueness, we already know that $r = |z| > 0$ is unique, so it remains to consider

$$x = r\cos(\theta_1) = r\cos(\theta_2), \qquad y = r\sin(\theta_1) = r\sin(\theta_2).$$

To translate the situation into an interval that we understand, pick $k_1, k_2 \in \mathbb{Z}$ so that $\theta_1 + 2\pi k_1, \theta_2 + 2\pi k_2 \in [-\pi, \pi)$. Then

$$\cos(\theta_1 + 2\pi k_1) = \cos(\theta_1) = \cos(\theta_2) = \cos(\theta_2 + 2\pi k_2).$$

Using the injectivity of the cosine function and considering cases as above, we find that $\theta_1 + 2\pi k_1 = \pm(\theta_2 + 2\pi k_2)$. If the sign is '+' we get $\theta_1 - \theta_2 = 2\pi(k_2 - k_1)$ and we are done, so suppose $\theta_1 + 2\pi k_1 = -(\theta_2 + 2\pi k_2)$. Then

$$\sin(\theta_1) = \sin(\theta_2) = \sin(\theta_2 + 2\pi k_1) = -\sin(\theta_1 + 2\pi k_1) = -\sin(\theta_1)$$

implies $\sin(\theta_1) = 0$. Therefore θ_1 is a multiple of 2π, which implies that $\theta_1 + 2\pi k_1 = -(\theta_2 + 2\pi k_2) = 0$ since these numbers were chosen in $[-\pi, \pi)$ and we have $2\pi\mathbb{Z} \cap [-\pi, \pi) = \{0\}$. Hence $\theta_1 - \theta_2 = 2\pi(k_2 - k_1)$. □

To get around the non-uniqueness of the argument in polar coordinates, we restrict θ to lie in a half-open interval of length 2π. Here is the most common convention.

Definition 1.9. The **principal argument** of a non-zero $z \in \mathbb{C}$ is the unique $\theta \in (-\pi, \pi]$ such that (1.16) holds, and we write $\arg(z) = \theta$.

Definition 1.10. The value of the **exponential function** at the complex number $z = x + i\theta$, where $x, \theta \in \mathbb{R}$, is defined as

$$e^{x+i\theta} = e^x\big(\cos(\theta) + i\sin(\theta)\big). \tag{1.17}$$

Proposition 1.8 implies that every complex number can be represented in **polar form**

$$z = re^{i\theta}. \tag{1.18}$$

The addition of complex numbers is the usual addition of vectors in $\mathbb{R}^2$. To visualize multiplication, the polar form is useful. Combining (1.2) and (1.17), we find that

$$\begin{aligned} e^{i\theta_1} \cdot e^{i\theta_2} &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ &\quad + i\big[\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)\big] \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

Here we have used the trigonometric addition formulas. Hence

$$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2} \implies z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}. \tag{1.19}$$

Complex multiplication adds the angles and multiplies the radii.

Proposition 1.11. *$e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$ for all $z_1, z_2 \in \mathbb{C}$. Moreover, we have $(e^z)^n = e^{nz}$ for all $z \in \mathbb{C}$, $n \in \mathbb{Z}$.*

Proof. The first assertion follows from (1.19) combined with the identity $e^{x_1}e^{x_2} = e^{x_1+x_2}$ for $x_1, x_2 \in \mathbb{R}$ from MA1005 Calculus. The second claim follows from this by induction. □

The polar form can be applied to the construction of n^{th} roots. For example, the **$\mathbf{n^{th}}$ root of unity** is $\zeta_n = e^{i\frac{2\pi}{n}}$ and satisfies

$$(\zeta_n)^n = (e^{i\frac{2\pi}{n}})^n = e^{2\pi i} = 1.$$

Proposition 1.12. *Every complex number $z \neq 0$ has an n^{th} root w satisfying $w^n = z$. If w is an n^{th} root of z, the set of all n^{th} roots of z is*

$$\{w, \zeta_n \cdot w, \zeta_n^2 \cdot w, \dots, \zeta_n^{n-1} \cdot w\}.$$

Proof. Write $z = re^{i\theta}$ and $w = se^{i\varphi}$ for $\theta, \varphi \in (-\pi, \pi]$ and $r, s > 0$. By the uniqueness of the polar form, the equation $w^n = z$ is equivalent to $s^n = r$ and $n\varphi = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Then $s = \sqrt[n]{r}$ is the unique positive n^{th} root from MA1005 Calculus. The solutions to $n\varphi = \theta + 2\pi k$ with $\varphi \in [0, 2\pi)$ are precisely

$$\varphi_k = \frac{\theta + 2\pi k}{n} \quad \text{for} \quad k = 0, \dots, n-1.$$

Hence $w_k = \sqrt[n]{r}e^{i\varphi_k}$, $k = 0, \dots, n-1$, are all the n^{th} roots of z. □

Todo

- Discussion of circlines
- Upper half-plane and other subsets of $\mathbb{C}$

Questions for further discussion

- The complex numbers are obtained by 'adjoining' a symbol i with $i^2 = -1$. If instead we would have adjoined a different symbol ϵ with $\epsilon^2 = 1$, would the set of elements $x + \epsilon y$ still define a field?
- The real numbers have a total order '$\leqslant$'. Why doesn't it make sense to extend this definition to the complex numbers?
- Describe geometrically the set $R_n = \{1, \zeta_n, \dots, (\zeta_n)^{n-1}\}$ of n^{th} roots of unity. Find a connection between R_n and the cyclic group $C_n = \{\overline{0}, \dots, \overline{n-1}\}$ of integers modulo n from MX3020 Group Theory.

Exercises

Exercise 1.1. Verify (1.3) and (1.4) straight from the definition (1.2).

Exercise 1.2. How many real solutions x does $x^2+1=0$ have? Show that the polynomial equation $z^2+1=0$ has exactly two solutions $z \in \mathbb{C}$.

Exercise 1.3. Sketch the position of the complex numbers $i, 1+i, \frac{3+2i}{4}$ in the plane.

Exercise 1.4. Express the following complex numbers z in the form $x+iy$ with $x, y \in \mathbb{R}$.

$$(1+i)2, \;\; (5+3i)(1+2i), \;\; (1-i)(2+3i), \;\; (1-i)i(1+i), \;\; \frac{2+i}{1-i}$$

Exercise 1.5. Express the following complex numbers z in the form $x+iy$ with $x, y \in \mathbb{R}$.

$$1/i, \;\; \frac{1}{1+i}, \;\; \frac{2+i}{1-i}, \;\; \frac{3+i}{3-i}$$

Exercise 1.6. Find the modulus and the conjugate of the following complex numbers.

$$2+i, \;\; i, \;\; 5-3i, \;\; \frac{1+i}{2+i}$$

Exercise 1.7. Describe the sets $A = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, $B = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leqslant 1\}$, $C = \{z \in \mathbb{C} \mid \mathrm{Re}((1+i)z) = 0\}$, and $A \cap B$ geometrically.

Exercise 1.8. Describe the set $D = \{z \in \mathbb{C} \mid z \cdot \overline{z} = 1\}$ geometrically. *Hint:* Write $z = re^{i\theta}$ in polar form.

Exercise 1.9. Show that $i = e^{i\pi/2}$ and $-1 = e^{i\pi}$.

Exercise 1.10. Express the following complex numbers z in the form $x+iy$ with $x, y \in \mathbb{R}$.

$$e^{i\pi/4}, \;\; e^{i\pi}, \;\; e^{i\frac{2i}{3}}$$

Exercise 1.11. Calculate i^{2021} and $(1+i)^{2021}$.

Exercise 1.12. Solve the equation $(1-i)^n - 2075 = 2021$ and find $n \in \mathbb{N}$.

Exercise 1.13. Prove that for $z \in \mathbb{R}$ we have

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}.$$

Use these equations to extend the definition of the functions $\cos(z), \sin(z)$ to complex arguments $z \in \mathbb{C}$. Find $z \in \mathbb{C}$ with $\sin(z) = 2$.
Hint: Put $w = e^{iz}$ and reduce to a quadratic equation.

Exercise 1.14. Show that $|z| = |-z|$ and $|\overline{z}| = |z|$. Prove also that $|\lambda z| = \lambda|z|$ for all $\lambda \geqslant 0$.

Exercise 1.15. Prove that $\overline{e^z} = e^{\overline{z}}$. Deduce that $|e^z| = e^{\operatorname{Re}(z)}$.

Exercise 1.16. Prove that $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

Exercise 1.17. Show that $|z+w|^2 = |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2$. Use this to determine the conditions on z, w for $|z+w| = |z| + |w|$ to hold.

Further resources

- Freitag–Busam [5, Chapter I.1] for additional exercises and historical background.
- `https://en.wikipedia.org/wiki/Complex_number` for overview and history
- `https://youtu.be/T647CGsuOVU` for some visualization

2 Complex-valued functions

Definition 2.1. A **complex function** $f\colon D \to \mathbb{C}$ is a map with **domain of definition** $D \subset \mathbb{C}$ and codomain the complex plane. Thus, f assigns to each $z = x + iy \in D$ in the domain a complex number

$$f(z) = u(z) + iv(z). \tag{2.1}$$

We call $u\colon D \to \mathbb{R}$ the **real part** and $v\colon D \to \mathbb{R}$ the **imaginary part** of the complex function f.

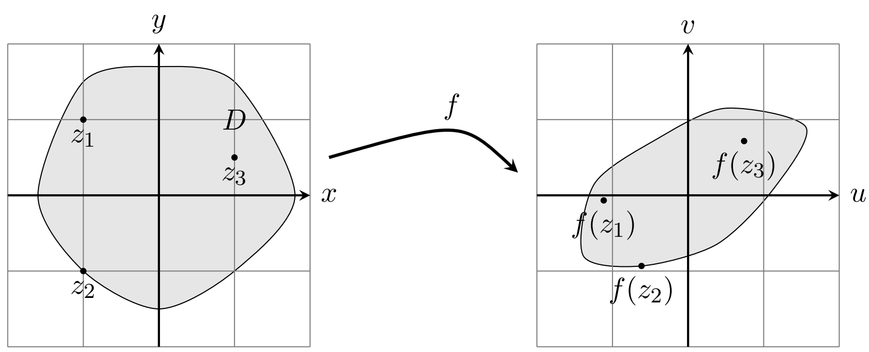

Figure 2.1: Schematic picture of a complex function

We can visualize complex functions as in Figure 2.1. In practice, most complex functions are defined by a formula.

Example 2.2. Let $a, b \in \mathbb{C}$ be fixed complex numbers. Then $f(z) = az + b$ is a complex function with domain $D = \mathbb{C}$, written $f : \mathbb{C} \to \mathbb{C}$. For instance,

$$f(z) = iz, \quad f(z) = 2z, \quad f(z) = z + 3.$$

Example 2.3. A **polynomial** is a complex function $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \tag{2.2}$$

with complex **coefficients** $a_n, \ldots, a_0 \in \mathbb{C}$ and with domain $D = \mathbb{C}$.

More generally, a **rational function** is a complex function for which there are polynomials f, g so that

$$h(z) = \frac{f(z)}{g(z)}. \tag{2.3}$$

The domain of h is $D = \{z \in \mathbb{C} \mid g(z) \neq 0\}$. However, if the polynomials f, g have a common factor, we can cancel that factor in the fraction (2.3) and regard h as a complex function on a larger domain.

For example, $h(z) = \frac{z+1}{z^2-1}$ has domain $\mathbb{C} \setminus \{\pm 1\}$, but we can rewrite the fraction as $h(z) = \frac{1}{z-1}$, which makes sense on the extended domain $\mathbb{C} \setminus \{+1\}$.

We have already met several complex functions in the previous section.

Example 2.4. The **exponential function** $\exp : \mathbb{C} \to \mathbb{C}$ is defined by $\exp(z) = e^z = e^x(\cos(y) + i \sin(y))$ and has domain $D = \mathbb{C}$.

Example 2.5. The **sine function** $\sin : \mathbb{C} \to \mathbb{C}$ and the **cosine function** $\cos : \mathbb{C} \to \mathbb{C}$ are the complex functions defined by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.4}$$

(See Exercise (1.13).)

Definition 2.6. We define the following subsets of the complex plane:

$$\begin{aligned} \mathbb{C}^\times &= \{z \in \mathbb{C} \mid z \neq 0\} && \textbf{punctured plane} \\ \mathbb{C}^- &= \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leqslant 0\} && \textbf{slit plane} \\ S &= \{z = x + iy \mid y \in (-\pi, \pi)\} && \textbf{principal strip} \end{aligned}$$

Example 2.7. The **principal branch** of the **complex logarithm** is the complex function

$$\log : \mathbb{C}^- \longrightarrow S \subset \mathbb{C} \tag{2.5}$$

defined by

$$\log(w) = \log(r) + i\theta \iff w = re^{i\theta}, r > 0, \theta \in (-\pi, \pi). \tag{2.6}$$

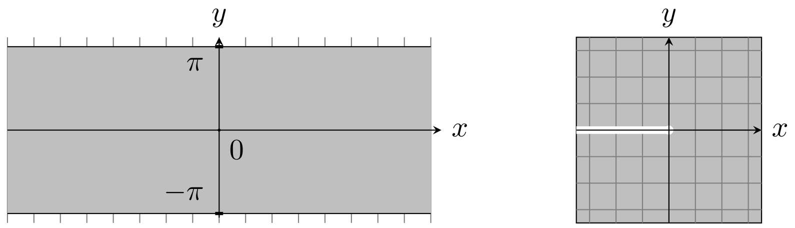

Figure 2.2: The principal strip S and the slit plane $\mathbb{C}^-$

Here $\log(r)$ denotes the (real) logarithm from MA1005 Calculus. In other words, the real part of $\log(w)$ is the logarithm of the modulus $r = |w|$ and the imaginary part of $\log(w)$ is the argument function $\arg(w)$ from Definition 1.9.

To evaluate (2.6), write w in polar coordinates, ensuring that $\theta \in (-\pi, \pi)$. For example, $i = e^{i\pi/2}$ and so $\log(i) = \log(1) + i\frac{\pi}{2} = i\frac{\pi}{2}$.

Proposition 2.8. (a) *The exponential function is surjective onto* $\mathbb{C}^\times$.

(b) *We have*

$$\exp(\log(w)) = w \quad (\forall w \in \mathbb{C}), \tag{2.7}$$
$$\log(\exp(z)) = z \quad (\forall z \in S). \tag{2.8}$$

Hence the restriction $\exp|_S$ *of the exponential to the principal strip is a bijection* $\exp|_S \colon S \to \mathbb{C}^-$ *onto the slit plane, with inverse* $\log(w)$.

Proof. (a) Recall from MA1005 Calculus that $e^x \neq 0$ for all $x \in \mathbb{R}$. As $|\exp(x+iy)| = e^x \neq 0$, the image of exp is contained in $\mathbb{C}^\times$. For proving that exp is onto $\mathbb{C}^\times$, recall that every non-zero number can be written in polar form $w = re^{i\theta}$, $\theta \in (-\pi, \pi]$. Then $\exp(z) = w$ for $z = \log(r) + i\theta$.
(b) Equations (2.7), (2.8) are straightforward to verify using (2.6). □

Since the **graph** $\Gamma(f) = \{(z, w) \in D \times \mathbb{C} \mid f(z) = w\}$ of a complex function is a subset of four-dimensional space, we cannot visualize complex functions as easily as real functions. We will now discuss some alternatives.

Image grid. A useful way to picture a complex function is to sketch its values on a grid G. The image grid $f(G)$ is a distorted version of the original grid which can be navigated easily using the grid lines. For example, to determine $f(1+2i)$, take one step in x-direction and two steps in y-direction on the distorted grid. Formally, let $G = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ be the **unit grid** and define the **image grid** as (see Figure 2.3)

$$f(G) = \{w = u + iv \in \mathbb{C} \mid \exists z \in G : f(z) = w\}.$$

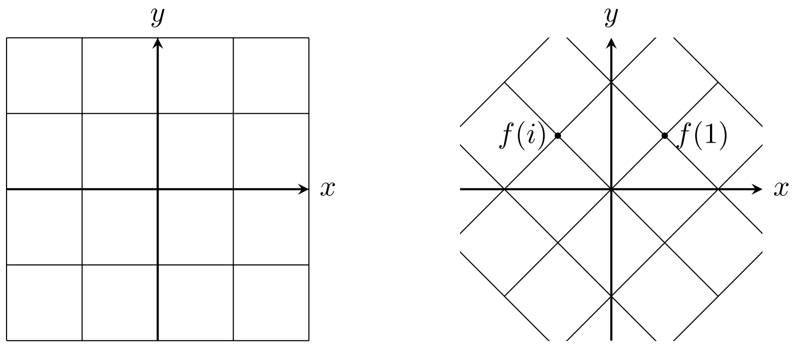

Figure 2.3: The unit grid G and the image grid $f(G)$ for $f(z) = \frac{1+i}{\sqrt{2}}z$

In practice, the image grid can often be described by finding a familiar equation that all of its members $u + iv$ satisfy. When this is not possible, a computer will help sketching an approximate image grid.

Example 2.9. Consider $f(z) = z^2$. Then

$$u = x^2 - y^2, \qquad v = 2xy. \tag{2.9}$$

To determine $f(G)$, first fix $x = \pm 1, \pm 2, \pm 3, \dots$ to be a non-zero integer. Using (2.9) we find $u = \frac{-1}{4x^2}v^2 + x^2$. This is a downward parabola in the (u, v)-plane rotated by 90 degrees with vertex at $(u, v) = (x^2, 0) = (1, 0), (4, 0), (9, 0), \dots$. Similarly, if we fix y to be a non-zero integer, then $u = \frac{1}{4y^2}v^2 - y^2$ is an upward parabola in the (u, v)-plane rotated by 90 degrees. For $x = 0$, we get $(u, v) = (-y^2, 0)$ which parameterizes the negative u-axis. Similarly, for $y = 0$ we get the positive u-axis. All this is summarized in Figure 2.4.

Example 2.10. Consider $f(z) = 1/z$. Then

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}. \tag{2.10}$$

Fixing $x = \pm 1, \pm 2, \dots$, we have $\left(u - \frac{1}{2x}\right)^2 + v^2 = \frac{1}{4x^2}$ (verify by substituting (2.10) into this equation). This is a circle of radius $\frac{1}{2x}$ and with center $(u, v) = (\frac{1}{2x}, 0)$. For $x = 0$, equations (2.10) become $(u, v) = (0, -1/y)$ which parameterizes the v-axis. Similarly, for $0 \neq y \in \mathbb{Z}$ we find $u^2 + \left(v + \frac{1}{2y}\right)^2 = \frac{1}{4y^2}$ and for $y = 0$ we obtain a parametrization of the u-axis. This is summarized in Figure 2.5 (and on the title page).

The previous example can be generalized.

Example 2.11. Rational functions (2.3) with $f(z) = az+b$ and $g(z) = cz+d$ affine linear, where $a, b, c, d \in \mathbb{C}$, are **Möbius transformations**. Thus

$$f(z) = \frac{az + b}{cz + d} \tag{2.11}$$

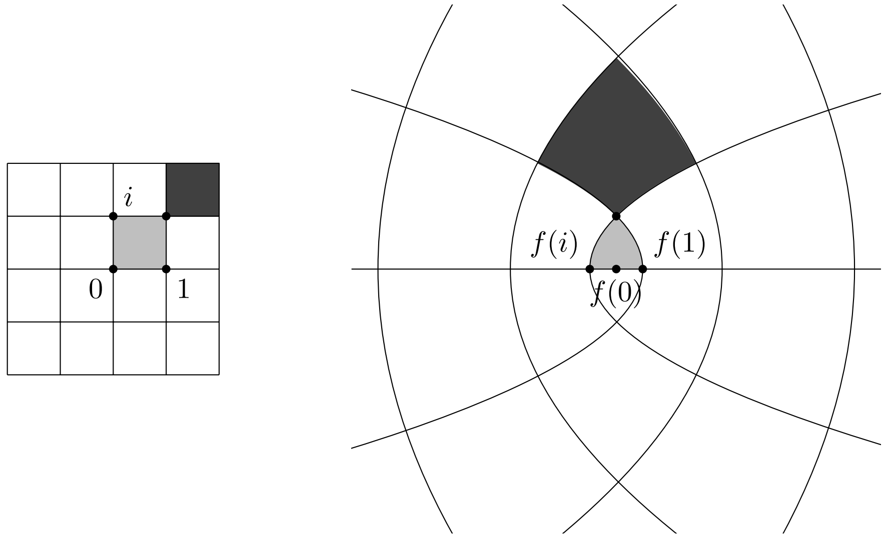

Figure 2.4: Unit grid and image grid of $f(z) = z^2$

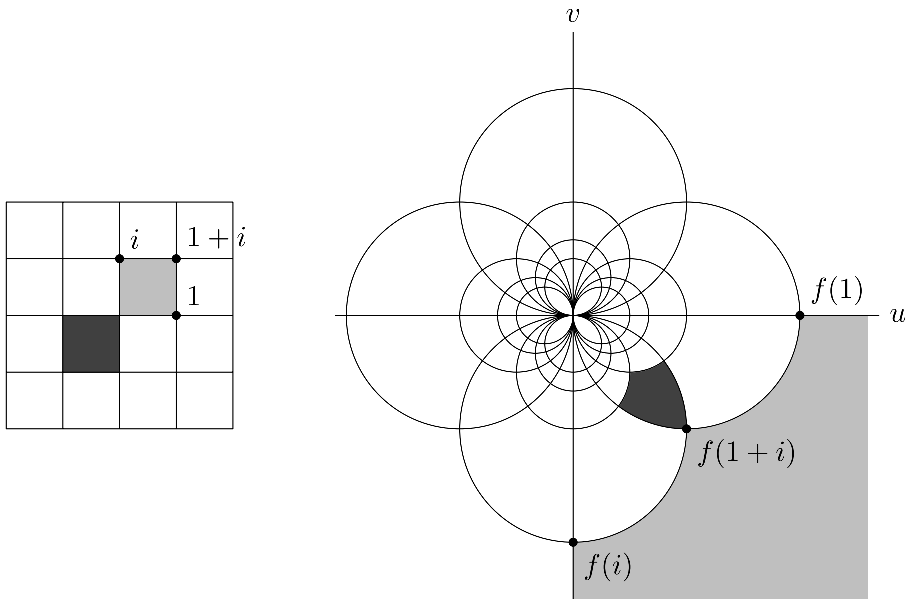

Figure 2.5: Unit grid and image grid of $f(z) = 1/z$. Notice that $z = 0$ gets sent to a 'point at infinity' that is imagined to surround the complex plane.

with domain $D = \mathbb{C} \setminus \{-d/c\}$ if $c \neq 0$ and domain $D = \mathbb{C}$ if $c = 0$. To exclude constant functions, we also assume that $ad - bc \neq 0$.

Domain colouring. We represent each unit complex number $e^{i\theta}$ by a color on the color wheel. The modulus r of an arbitrary complex number $re^{i\theta}$ will be represented by the lightness of the color. This assigns a unique color to each complex number, see Figure 2.6. Pure white is never used and would correspond to infinity. Pure black corresponds to the origin.

This is less useful for calculations by hand but generates artistic images using a computer.

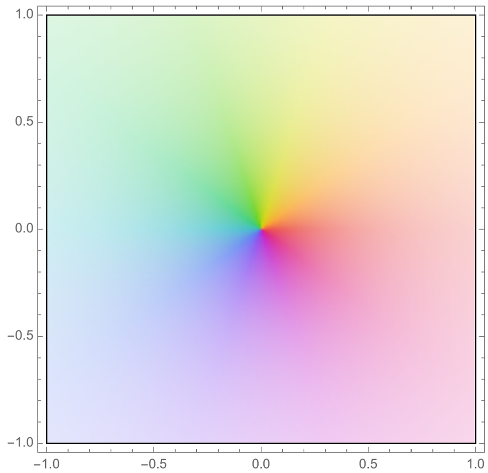

Figure 2.6: Representing complex numbers z by color

This can be used for visualizing complex functions. Draw each point z in the domain of $w = f(z)$ using the color for w.

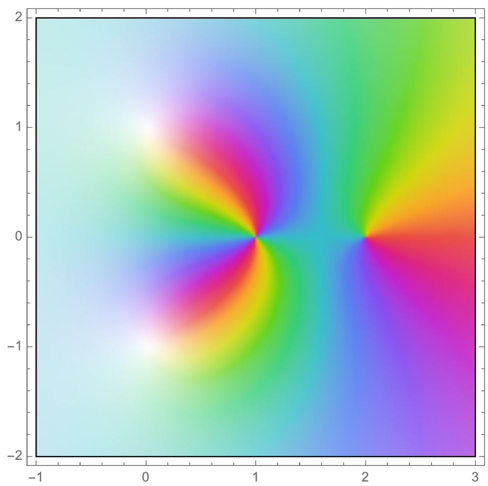

Figure 2.7: Domain colouring of $f(z) = \frac{(z-2)(z-1)^2}{z^2+1}$. Notice that near $z = \pm i$, where f is undefined, the function tends to infinity (pure white). The zeros of f at $z = 1, 2$ can also be seen.

3-dimensional graphs. Another approach is to plot the 3-dimensional graph of any of the following real-valued functions $D \to \mathbb{R}$

$$u, v, |f| = \sqrt{u^2 + v^2}.$$

Again, the missing information can be color-coded (see Figure 2.8).

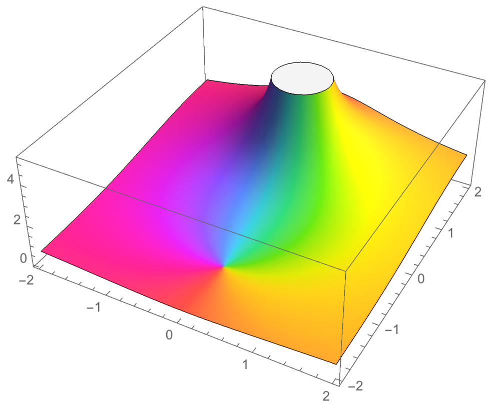

Figure 2.8: The graph of $|f(z)|$ colored by $\arg(z)$

Questions for further discussion

- What are the functions in Example 2.2 geometrically?
- Does (2.8) remain valid for all $z \in \mathbb{C}$? What is the correct modification?
- The two zeros in Figure 2.7 have a slightly different character. What is the difference between the zeros that might account for this?
- In Figures 2.4 and 2.5 almost all the image gridlines meet at a right angle. The only exception is at $f(0)$ in Figure 2.4. Use polar coordinates to explain this behavior of $f(z) = z^2$ at $z = 0$.
- Is it always possible to extend the domain of definition of a complex function? Is this always sensible?
- Can you think of other branches of the logarithm?

Exercises

Exercise 2.1. Let $\log(z)$ be the principal branch of the logarithm. Compute

$$\log(2i), \quad \log(1 + i), \quad \log(-3i), \quad \log(5).$$

Exercise 2.2. Show that $\log(zw) = \log(z) + \log(w)$ provided $z, w, zw \in \mathbb{C}^-$.

Exercise 2.3. Describe the following complex functions geometrically.

$$f(z) = 3z, \quad f(z) = iz, \quad f(z) = \frac{(1+i)}{\sqrt{2}}z$$

Exercise 2.4. Determine a domain of definition for the following complex functions.

$$f(z) = \frac{1}{z}, \quad f(z) = \frac{1+z}{z-1}, \quad f(z) = \frac{z^2-4}{z^2+2z}, \quad f(z) = \frac{1}{\exp(z)}.$$

Exercise 2.5. Determine the domain of definition for $f(z) = \frac{1}{\sin(z)}$.

Exercise 2.6. Prove that the composition $f \circ f'$ of two Möbius transformations is again a Möbius transformation. If we associate to f, f' the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{C}),$$

show that $f \circ f'$ is associated to the matrix product AA'.

Exercise 2.7. Draw the image grid for $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

Further resources

- `https://youtu.be/NtoIXhUgqSk` 5 ways to visualize a complex function
- `https://youtu.be/0z1fIsUNhO4` and `https://www-users.cse.umn.edu/~arnold/moebius/` Animation of Möbius transformations

3 Topology of the complex plane

In this section we explain that most of the facts about limits, series, and continuity carry over from real analysis essentially without change.

The modulus of complex numbers defines a **distance** $d(z, w) = |z - w|$ on the plane (this is the usual Euclidean distance), which determines the following standard terminology for metric spaces.

Definition 3.1. The **open disc** of radius $0 \leqslant r \leqslant +\infty$ centered at $z_0 \in \mathbb{C}$ is

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}. \tag{3.1}$$

The **closed disc** $\overline{D}_r(z_0)$ is the set of all $z \in \mathbb{C}$ with $|z - z_0| \leqslant r$.

Definition 3.2. A subset $O \subset \mathbb{C}$ is **open** if for every point $z_0 \in O$ there exists $r > 0$ such that $D_r(z_0) \subset O$ (see Figure 3.1). A subset $C \subset \mathbb{C}$ is called **closed** if the complement $O = \mathbb{C} \setminus C$ is an open subset.

A subset $B \subset \mathbb{C}$ is **bounded** if $B \subset D_r(0)$ for some $0 < r < \infty$.

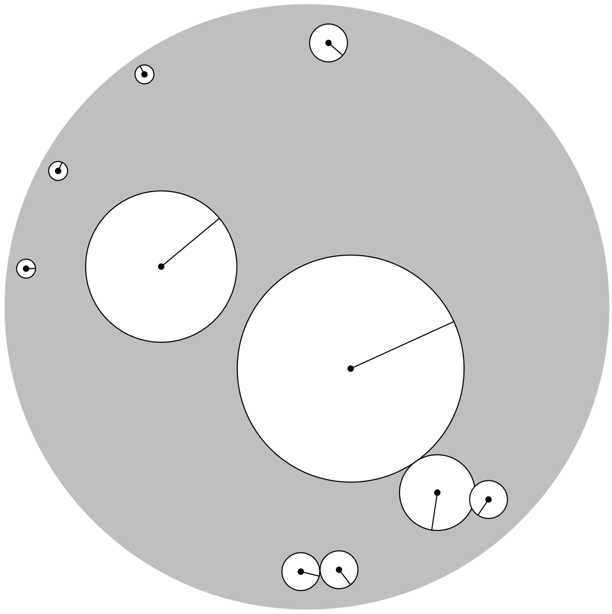

Figure 3.1: The gray open disc is an open subset of $\mathbb{C}$.

Remark 3.3. The above notion of open set determines a topology on $\mathbb{C}$.

Definition 3.4. A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ has the **limit** $\zeta \in \mathbb{C}$ (or is **convergent** to ζ), written $\lim_{n\to\infty} z_n = \zeta$, if

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0 : |z_n - \zeta| < \epsilon.$$

Equivalently, $\lim_{n\to\infty} |z_n - \zeta| = 0$. We call $(z_n)_{n\in\mathbb{N}}$ a **Cauchy sequence** if

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m \geqslant n_0 : |z_n - z_m| < \epsilon.$$

The same argument as in real analysis shows that the limit w is unique and that every convergent sequence is a Cauchy sequence. The converse is also true by the completeness of real numbers.

Proposition 3.5. *For a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$, the following are equivalent:*

(a) *There exists $\zeta \in \mathbb{C}$ such that $\zeta = \lim_{n\to\infty} z_n$.*

(b) *$(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. To prove (b) $\Longrightarrow$ (a) write $z_n = x_n + iy_n$ and notice that

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} \geqslant |x_n - x_m|,\ \ |y_n - y_m|$$

implies that both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$. By the completeness of $\mathbb{R}$, these sequences have limits $\chi, \upsilon \in \mathbb{R}$. Set $\zeta = \chi + i\upsilon$ and pick $n_0 \in \mathbb{N}$ such that $|x_n - \chi| < \frac{\epsilon}{\sqrt{2}}$ and $|y_n - \upsilon| < \frac{\epsilon}{\sqrt{2}}$ for all $n \geqslant n_0$. Then

$$|z_n - \zeta| = \sqrt{(x_n - \chi)^2 + (y_n - \upsilon)^2} < \sqrt{\epsilon^2/2 + \epsilon^2/2} = \epsilon$$

for all $n \geqslant n_0$. The converse, (a) $\Longrightarrow$ (b), is left as an exercise. $\square$

The advantage of Cauchy sequences is that one does not need to know the value of the limit in advance.

Definition 3.6. A **series** of complex numbers $(z_k)_{k\in\mathbb{N}}$ has the **limit** ζ, written $\zeta = \sum_{k=0}^{\infty} z_k$, if the sequence $\big(w_n = \sum_{k=0}^{n} z_k\big)_{n\in\mathbb{N}}$ converges to ζ. We call a series **absolutely convergent** if the series $\sum_{k=0}^{\infty} |z_k|$ is convergent.

As in real analysis, the Cauchy criterion implies that every absolutely convergent series is convergent. Absolutely convergent series may be rearranged and orders of summation may be exchanged.

Although $\infty \notin \mathbb{C}$, it will be convenient to define $\lim_{n\to\infty} z_n = \infty$ to mean that the sequence $(z_n)_{n\in\mathbb{N}}$ eventually leaves every disk. Symbolically,

$$\forall N > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geqslant n_0 : |z_n| > N.$$

We call $\mathbb{C} \cup \{\infty\}$ the **extended complex plane**.

Definition 3.7. A point $\zeta \in \mathbb{C} \cup \{\infty\}$ is in the **closure** of $D \subset \mathbb{C}$ if there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim_{n\to\infty} z_n = \zeta$.

Let $f\colon D \to \mathbb{C}$ be a complex function and let $\zeta \in \mathbb{C} \cup \{\infty\}$ be in the closure of D. The function $f(z)$ has the **limit** $w \in \mathbb{C} \cup \{\infty\}$ as $z \to \zeta$, written $\lim_{z\to\zeta} f(z) = w$, if for *every* sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim_{n\to\infty} z_n = \zeta$ we have $\lim_{n\to\infty} f(z_n) = w$.

A complex function $f\colon D \to \mathbb{C}$ is **continuous at** $\zeta \in D$ if $\lim_{z\to\zeta} f(z) = f(\zeta)$. We call f **continuous on** D if f is continuous at every $\zeta \in D$.

Questions for further discussion

- Explain the difference between the notions 'limit of a function', 'limit of a sequence', and 'limit of a series'.

Exercises

Exercise 3.1. For which $z \in \mathbb{C}$ do the following limits exist?

$$\lim_{n\to\infty} n^{1/n} z, \quad \lim_{n\to\infty} z^n, \quad \lim_{n\to\infty} \frac{z^n}{n}, \quad \lim_{n\to\infty} \frac{z^n}{n!}, \quad \lim_{n\to\infty} \frac{z^n}{n^n}, \quad \lim_{n\to\infty} n! z.$$

Exercise 3.2. Show that every convergent sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is bounded.

Exercise 3.3. Let $\sum_{k=0}^{\infty} z_k$ be a convergent series of complex numbers. Show that $\lim_{k\to\infty} z_k = 0$.

Exercise 3.4. Recall the ratio and root test for series of real numbers.

Exercise 3.5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a complex function. Show that

$$\lim_{z\to\infty} f(z) = w \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w.$$

Exercise 3.6. The Riemann sphere is $\mathbb{S} = \{(a,b,c) \in \mathbb{R}^3 \mid a^2+b^2+c^2 = 1\}$. Show that the stereographic projection

$$F\colon \mathbb{S} \longrightarrow \mathbb{C} \cup \{\infty\}, \quad F(a,b,c) = \begin{cases} \frac{a+ib}{1-c} & \text{if } c \neq 1, \\ \infty & \text{if } c = 1. \end{cases}$$

is a bijection between the Riemann sphere and the extended complex plane. Find a formula for the inverse function.

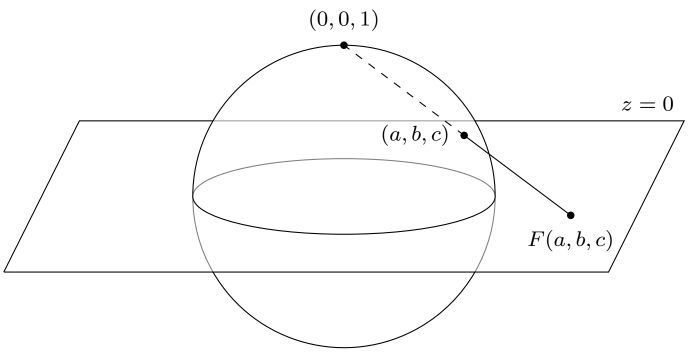

Figure 3.2: Stereographic projection from the north pole

Further resources

Chapter II

Differentiation and contour integrals

4 Holomorphic functions

The following definition is a key concept for this course.

Definition 4.1. A complex function $f\colon U \to \mathbb{C}$ with domain an open set $U \subset \mathbb{C}$ is **complex differentiable** at a point $z_0 \in U$ if the limit

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.1}$$

exists. We call f **holomorphic** on U if f is complex differentiable at every $z_0 \in U$. The set of all holomorphic functions on U is denoted $\mathcal{O}(U)$. A function that is holomorphic[1] on $U = \mathbb{C}$ is **entire**.

This definition is structurally the same as for real functions. For example, f is continuous at z_0 since multiplying (4.1) by $0 = \lim_{z \to z_0}(z - z_0)$ gives

$$0 = \lim_{z \to z_0} f(z) - f(z_0) \implies f(z_0) = \lim_{z \to z_0} f(z).$$

Other formal proofs carry over as well and show the following.

Proposition 4.2. *Let $f, g\colon U \to \mathbb{C}$ be complex differentiable at z_0. Then:*

(a) *$f + g$ is complex differentiable at z_0 with*

$$(f + g)'(z_0) = f'(z_0) + g'(z_0). \tag{4.2}$$

(b) *fg is complex differentiable at z_0 with*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{4.3}$$

[1] From Greek *holos* 'whole, complete' and *morphē* 'form, shape'

(c) *If $g'(z_0) \neq 0$, then f/g is complex differentiable at z_0 with*

$$(f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{4.4}$$

Proof. Exercise, using the algebra of limits. □

Example 4.3. Directly from (4.1) we find that the constant function $f(z) = c$ is entire with $f'(z) = 0$. The identity function $f(z) = z$ is entire with $f'(z) = 1$. Proposition 4.2 and induction then imply that all polynomials

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

are entire with

$$f'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \ldots + a_1.$$

It then follows from Proposition 4.2(c) that rational functions are holomorphic on their domain of definition.

Proposition 4.4. *Let $f\colon U \to \mathbb{C}$, $g\colon V \to \mathbb{C}$ be complex functions with $f(U) \subset V$, and U, V open. Suppose that f is complex differentiable at $z_0 \in U$ and that g is complex differentiable at $w_0 = f(z_0) \in V$.*

Then $g \circ f$ is complex differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(w_0) f'(z_0). \tag{4.5}$$

Proof. We can equivalently rewrite (4.1) as

$$f(z_0 + h) = f(z_0) + h f'(z_0) + h\rho(h), \qquad \lim_{h \to 0} \rho(h) = 0, \tag{4.6}$$

where the function $\rho(h)$, $h \neq 0$, is defined by this equation. Similarly,

$$g(w_0 + k) = g(w_0) + k g'(w_0) + k\sigma(k), \qquad \lim_{k \to 0} \sigma(k) = 0. \tag{4.7}$$

Then

$$\frac{(g \circ f)(z_0 + h) - (g \circ f)(z_0)}{h} \overset{(4.6)}{=} \frac{g\big(w_0 + \overbrace{h f'(z_0) + h\rho(h)}^{k=}\big) - g(w_0)}{h}$$

$$\overset{(4.7)}{=} f'(z_0) g'(w_0) + \big[\rho(h) g'(w_0) + (f'(z_0) + \rho(h))\, \sigma(h f'(z_0) + h\rho(h))\big],$$

and the square bracket tends to zero as $h \to 0$, using the chain rule for limits to see that $\lim_{h \to 0} \sigma(h f'(z_0) + h\rho(h)) = 0$. □

One should view (4.6) as a short Taylor series expansion: near the point z_0 the function f is given to zeroth order by a constant $f(z_0)$, with first order correction given by a *complex linear* function $h \mapsto hf'(z_0)$, plus a higher order term $h\rho(h)$ which vanishes infinitesimally (after dividing by h).

We next discuss how complex differentiability compares to real differentiability. Regard a complex function as a real multivariable map

$$f = (u, v) \colon U \longrightarrow \mathbb{R}^2, \qquad U \subset \mathbb{R}^2 \text{ open set.}$$

Recall that f is **real differentiable** at $z_0 = (x_0, y_0)$ if there exists a matrix $J_f(z_0) \in \mathrm{M}_{2\times 2}(\mathbb{R})$, called the **Jacobian**, so that (4.1) holds with the complex multiplication $hf'(z_0)$ replaced by the matrix vector multiplication $J_f(z_0)h$. Real differentiability at z_0 implies that the partial derivatives at z_0 exist and

$$J_f(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x}(z_0) & \frac{\partial u}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) & \frac{\partial v}{\partial y}(z_0) \end{pmatrix}. \tag{4.8}$$

Recall that the converse requires the additional assumptions that the partial derivatives exist *in a neighborhood* of z_0 and are *continuous* at z_0.

Proposition 4.5. *For a complex function $f = u + iv$ the following are equivalent:*

(a) *f is complex differentiable at z_0*

(b) *f is real differentiable at z_0 and the **Cauchy–Riemann equations***

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0), \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0), \tag{4.9}$$

hold.

In this case, $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Proof. The two definitions are identical except that the complex multiplication $f'(z_0)h$ by $h \in \mathbb{C}$ is replaced by the matrix vector multiplication $J_f(z_0)h$ by $h \in \mathbb{R}^2$. Our task is to compare these two notions. By Proposition 1.4 $J_f(z_0)$ is $\mathbb{C}$-linear if and only if (4.9) holds. □

Example 4.6. For the exponential function, $u = e^x \cos(y)$, $v = e^x \sin(y)$. Then the partial derivatives exist

$$\frac{\partial u}{\partial x} = e^x \cos(y), \quad \frac{\partial u}{\partial y} = -e^x \sin(y), \quad \frac{\partial v}{\partial x} = e^x \sin(y), \quad \frac{\partial v}{\partial y} = e^x \cos(y),$$

are continuous functions on $\mathbb{R}^2$, and satisfy the Cauchy–Riemann equations. It follows that exp is entire with

$$\exp'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \exp(z). \tag{4.10}$$

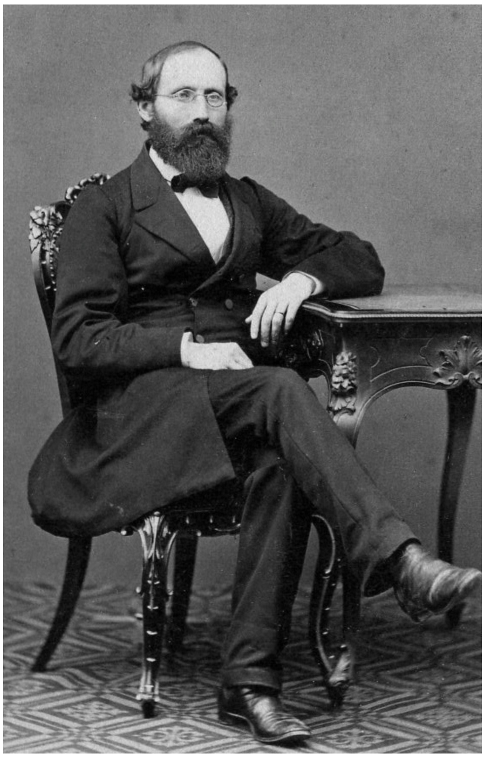

Figure 4.1: Bernhard Riemann. Familienarchiv Thomas Schilling, ca. 1862. WikiMedia Commons. Public Domain

Example 4.7. The **hyperbolic cosine** $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and the **hyperbolic sine** function $\sinh(z) = \frac{e^z-e^{-z}}{2}$ are entire with

$$\sinh'(z) = \cosh(z), \qquad \cosh'(z) = \sinh(z).$$

Since $\cosh(z) = 0 \iff e^{2z} = -1 \iff z \in \frac{\pi i}{2} + \pi i\mathbb{Z}$ the **hyperbolic tangent** $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ is a holomorphic function

$$\tanh\colon \mathbb{C} \setminus \left(\frac{\pi i}{2} + \pi i\mathbb{Z}\right) \longrightarrow \mathbb{C},$$

with

$$\tanh'(z) = \frac{\cosh(z)^2 - \sinh(z)^2}{\cosh(z)^2} = 1 - \tanh(z)^2. \tag{4.11}$$

Similarly, the **hyperbolic cotangent** $\coth(z) = \frac{\cosh(z)}{\sinh(z)}$ is a holomorphic function $\coth\colon \mathbb{C} \setminus \pi i\mathbb{Z} \to \mathbb{C}$ with

$$\coth'(z) = 1 - \coth(z)^2. \tag{4.12}$$

Example 4.8. The trigonometric sine $\sin(z) = -i\sinh(iz)$ and the cosine $\cos(z) = \cosh(iz)$ are entire with $\sin'(z) = \cos(z)$, $\cos'(z) = -\sin(z)$. The tangent $\tan(z) = \frac{\sin(z)}{\cos(z)}$ is a holomorphic function on $\mathbb{C} \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right)$ with $\tan'(z) = 1 + \tan(z)^2$. The cotangent $\cot(z) = \frac{\cos(z)}{\sin(z)}$ is a holomorphic function on $\mathbb{C} \setminus \pi\mathbb{Z}$ with $\cot'(z) = 1 - \cot(z)^2$.

Example 4.9. $f(z) = \bar{z} = x - iy$ has all partial derivatives of all orders, but is *not* complex differentiable at any $z_0 \in \mathbb{C}$ since $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$.

Theorem 4.10. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ with $f'(z_0) \neq 0$. Then there exist open sets $V, W \subset \mathbb{C}$ with $z_0 \in V \subset U$ and $w_0 = f(z_0) \in W$ with the property that the restriction $f|_V$ becomes a bijection $V \to W$ with holomorphic inverse function $(f|_V)^{-1}\colon W \to V$ and*

$$\frac{\partial (f|_V)^{-1}}{\partial w}(w) = \frac{1}{f'\big((f|_V)^{-1}(w)\big)}. \tag{4.13}$$

Proof. As in the proof of Proposition 4.5, we view f as a differentiable real multivariable map $U \to \mathbb{R}^2$. Since $f'(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial u}{\partial y}(z_0) \neq 0$ we have

$$\begin{aligned} \det J_f(z_0) &\overset{(4.8)}{=} \frac{\partial u}{\partial x}(z_0)\frac{\partial v}{\partial y}(z_0) - \frac{\partial v}{\partial x}(z_0)\frac{\partial u}{\partial y}(z_0) \\ &\overset{(4.9)}{=} \left(\frac{\partial u}{\partial x}(z_0)\right)^2 + \left(\frac{\partial u}{\partial y}(z_0)\right)^2 \neq 0. \end{aligned}$$

We may thus apply the inverse function theorem from multivariable calculus. This gives open sets with $z_0 \in V \subset U, w_0 \in W$ such that $f|_V\colon V \to W$ is a bijection and $(f|_V)^{-1}\colon W \to V$ is real differentiable. In particular, $\det J_f(z) \neq 0$ for all $z \in V$ (chain rule). Let $w \in W$ and $z = f^{-1}(w)$. Then

$$J_{(f|_V)^{-1}}(w) = J_f(z)^{-1} = \frac{1}{\det J_f(z)} \begin{pmatrix} \frac{\partial v}{\partial y}(z) & -\frac{\partial u}{\partial y}(z) \\ -\frac{\partial v}{\partial x}(z) & \frac{\partial u}{\partial x}(z) \end{pmatrix},$$

so the Cauchy–Riemann equations are satisfied at every $w \in W$ for the inverse function, so $(f|_V)^{-1}$ is holomorphic. Finally, (4.13) follows by applying the chain rule (4.5) to $f|_V \circ (f|_V)^{-1} = \mathrm{id}_W$ which gives

$$(f|_V)'\big((f|_V)^{-1}(w)\big) \cdot \frac{\partial (f|_V)^{-1}}{\partial w}(w) = 1. \qquad \square$$

Remark 4.11. Our proof is a simple application of the ordinary inverse function theorem. We will see two different proofs later in the course.

Example 4.12. Recall from Proposition 2.8 that $\exp|_S\colon S \to \mathbb{C}^-$ is a bijection with the logarithm as inverse function. As $\exp'(z) = \exp(z) \neq 0$, we conclude from Theorem 4.10 that $\log\colon \mathbb{C}^- \to S$ is holomorphic with

$$\log'(z) = 1/z. \tag{4.14}$$

Questions for further discussion

- $x \mapsto x^3$ is a real differentiable bijection whose inverse function is not differentiable at $y = 0$. What about the complex analogue $z \mapsto z^{1/3}$? Which branch?

Exercises

Exercise 4.1. Let $f\colon V \to W$ be a bijection of open sets $V, W \subset \mathbb{C}$. Assume that f is complex differentiable at z_0 and that f^{-1} is complex differentiable at $w_0 = f(z_0)$. Show that $f'(z_0) \neq 0$. (The same result holds for real differentiable maps to show $\det J_f(z_0) \neq 0$.)

Exercise 4.2. Show that every holomorphic function is harmonic. Wirtinger, write Laplace using complex derivatives.

Exercise 4.3. Find subdomains where $\sin, \cos, \sinh, \cosh, \tan, \tanh$ are injective. Compute the derivative of an inverse using the chain rule. Deduce an explicit formula.

Exercise 4.4. Show that the set $\mathcal{O}(U)$ of holomorphic functions on U becomes a ring under the operations of point-wise addition and multiplication:

$$(f+g)(z) = f(z) + g(z), \qquad (fg)(z) = f(z)g(z).$$

5 Power series

Definition 5.1. A **formal power series** is an expression of the form

$$P = \sum_{n=0}^{\infty} a_n T^n \tag{5.1}$$

with **coefficents** $a_n \in \mathbb{C}$. As no convergence is required, this is really just a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers.

We call a_0 the **constant term** of P. The **order** of P is the smallest $n \in \mathbb{N}$ such that $a_n \neq 0$.

Example 5.2. Every polynomial is a formal power series.

Definition 5.3. Let $\mathbb{C}[\![T]\!]$ be the set of all formal power series. Define the addition and multiplication of $P = \sum_{n=0}^{\infty} a_n T^n$, $Q = \sum_{n=0}^{\infty} b_n T^n \in \mathbb{C}[\![T]\!]$ by

$$P + Q = \sum_{n=0}^{\infty} (a_n + b_n) T^n, \tag{5.2}$$

$$PQ = \sum_{n=0}^{\infty} c_n T^n, \quad c_n = \sum_{i+j=n} a_i b_j. \tag{5.3}$$

Proposition 5.4. (a) *These operations make* $\mathbb{C}[\![T]\!]$ *a commutative ring with unit* 1, *the series with constant term* 1 *and all higher coefficients zero.*

(b) $P \in \mathbb{C}[\![T]\!]$ *has a multiplicative inverse* $\iff$ *the constant term of* P *is non-zero.*

Proof. (a) is straightforward and left as an exercise.

(b) Having an inverse amounts to the existence of $Q = \sum_{n=0}^{\infty} b_n T^n$ such that $PQ = 1$. By (5.3) this means

$$a_0 b_0 = 1 \qquad \sum_{i+j=n} a_i b_j = 0 \quad (\forall n > 0). \tag{5.4}$$

The first equations shows that a_0 is invertible, proving '$\implies$'. For '$\impliedby$', the idea is to use (5.4) to construct the coefficients b_n of the inverse inductively. Of course, $b_0 = a_0^{-1}$. For $n = 1$, (5.4) gives $a_0 b_1 + a_1 b_0 = 0$ so

$$b_1 = -a_0^{-2} a_1.$$

For $n = 2$, $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$. Inserting the known b_0, b_1 implies

$$b_2 = a_0^3 (a_1^2 - a_0 a_2).$$

For general n, given $b_0, \dots, b_{n-1}$ we rearrange (5.4) as $b_n = a_0^{-1} \sum_{j=0}^{n-1} a_i b_j$, which defines b_n by recursion. □

It is unnecessary to remember (5.4). It is enough to recall the ansatz $PQ = 1$ and the strategy of computing the coefficients of Q inductively.

Example 5.5. Let $P = 1 - T$. Then $P^{-1} = \sum_{n=0}^{\infty} T^n$ is the geometric series. We verify $(1 - T) \sum_{n=0}^{\infty} T^n = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = 1$ (telescope sum).

Definition 5.6. Fix a point $z_0 \in \mathbb{C}$ at which the power series (5.1) is to be **centered**. The **domain** $D(P, z_0)$ is the set of all $z \in \mathbb{C}$ such that the series

$$P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

of complex numbers converges. This defines a complex function

$$D(P, z_0) \longrightarrow \mathbb{C}, \quad z \longmapsto P(z). \tag{5.5}$$

Example 5.7. Let $z_0 = 0$. The geometric series $P = \sum_{n=0}^{\infty} T^n$ converges for all $|z| < 1$ since the partial sums $P_n(z) = \frac{z^{n+1}-1}{z-1}$ tend to $\frac{1}{1-z}$ as $n \to \infty$. The geometric series diverges for all $|z| \geqslant 1$, since in this case $(z^n)_{n \in \mathbb{N}}$ is not a null sequence. Hence $D(P, 0) = D_1(0)$.

Recall the following concept from analysis.

Definition 5.8. For a complex function $f\colon D \to \mathbb{C}$, the **uniform norm** is

$$\|f\|_{\infty,D} = \sup_{z\in D} |f(z)|. \tag{5.6}$$

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ is **uniformly convergent** on D if

$$\|f - f_n\|_{\infty,D} \to 0 \quad \text{as} \quad n \to \infty.$$

A series $\sum_{n=0}^{\infty} f_n(z)$ is a **uniformly convergent series** on D if the sequence of **partial sums** $P_n(z) = \sum_{k=0}^{n} f_k(z)$ converges uniformly.

Uniform convergence on a subset $C \subset D$ refers to the uniform convergence of the functions restricted to C.

Theorem 5.9. *For every formal power series P and center z_0 there is a unique **radius of convergence** $0 \leqslant \rho(P, z_0) \leqslant +\infty$ such that*

$$D_\rho(z_0) \subset D(P, z_0) \subset \overline{D}_\rho(z_0). \tag{5.7}$$

In fact,

$$\rho = \sup\left\{ |z - z_0| \geqslant 0 \;\middle|\; (|a_k||z - z_0|^k)_{k\in\mathbb{N}} \text{ is a bounded sequence} \right\}. \tag{5.8}$$

Moreover, $P(z)$ converges absolutely and uniformly on every smaller disk $D_r(z_0)$ with $0 < r < \rho$ and diverges at every point $z \in \mathbb{C}$ with $|z| > \rho$.

Proof. For uniqueness, suppose $D_{\rho_k}(z_0) \subset D(P, z_0) \subset \overline{D}_{\rho_k}(z_0)$ for $k = 1, 2$ and $\rho_1 \neq \rho_2$, say $0 \leqslant \rho_1 < \rho_2$. Pick $z \in \mathbb{C}$ with $\rho_1 < |z - z_0| < \rho_2$. Then $z \in D_{\rho_2}(z_0) \subset D(P, z_0) \subset \overline{D}_{\rho_1}(z_0)$, so $|z - z_0| \leqslant \rho_1$, a contradiction.

We only prove the last part for ρ defined by (5.8), as this implies (5.7). For $z \in \mathbb{C}$ with $\rho < |z - z_0|$ the terms $a_k(z - z_0)^k$ do not tend to zero (are even unbounded), so the series $P(z)$ diverges.

Let $0 < r < \rho$. Using that ρ is the *least* upper bound, there exists $w \in \mathbb{C}$ with $r < |w - z_0| < \rho$ and $|a_k||w - z_0|^k < B$ bounded. For all $z \in D_r(z_0)$,

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n| = \sum_{n=0}^{\infty} \left| a_n(w - z_0)^n \underbrace{\left(\frac{z - z_0}{w - z_0}\right)}_{q=}{}^{n} \right| \leqslant B \sum_{n=0}^{\infty} q^n. \tag{5.9}$$

Since $|q| < 1$, the geometric series on the right converges. Hence the comparison test implies that $P(z)$ converges absolutely. As for uniform convergence, let P_n be the n-th partial sum. As in (5.9), for all $z \in D_r(z_0)$

$$|P(z) - P_n(z)| = \left| \sum_{k=n}^{\infty} a_k(z - z_0)^k \right| \leqslant B \sum_{k=n}^{\infty} q^k.$$

The right hand side is independent of z, so $\|P - P_n\|_{\infty, D_r(z_0)} \xrightarrow{n\to\infty} 0$. $\square$

Remark 5.10. The domain $D(P)$ of a formal power series is roughly a disk of radius ρ, with uncertain behavior on the boundary circle $|z - z_0| = \rho$. Determining the behavior on the boundary can be very difficult. On the other hand, the radius of convergence is easy to compute. If you find $z \in \mathbb{C}$ for which $P(z)$ converges (for example, using the ratio or the root test), you can deduce $|z - z_0| < \rho$. If $P(z)$ diverges at $z \in \mathbb{C}$, you know $|z - z_0| \geqslant \rho$.

Example 5.11 (optional)**.** For $P = \sum_{n=1}^{\infty} \frac{z^n}{n}$ we have $\rho = 1$ by (5.8). We investigate the behavior on the boundary circle $|z| = 1$. For $z = 1$, this is the divergent harmonic series. We claim that P converges for all other boundary points, so $D(P; 0) = \overline{D}_1(0) \setminus \{1\}$. Notice the telescope sum

$$\begin{aligned}(z-1)\sum_{k=1}^{n} \frac{z^k}{k} &= \sum_{k=2}^{n+1} \frac{z^k}{k-1} - \sum_{k=1}^{n} \frac{z^k}{k} \\ &= \frac{z^{n+1}}{n} - z + \sum_{k=2}^{n} \left(\frac{z^k}{k-1} - \frac{z^k}{k} \right) \\ &= \frac{z^{n+1}}{n} - z + \sum_{k=2}^{n} \frac{z^k}{k(k-1)}.\end{aligned}$$

The series $\sum_{k=2}^{\infty} \frac{z^k}{k(k-1)}$ is (absolutely) convergent. It follows that the left hand sequence of partial converges as $n \to \infty$. Dividing by $z - 1 \neq 0$ implies that P converges.

Corollary 5.12. *The restriction of the complex function* (5.5) *to* $D_\rho(z_0) \subset D(P, z_0)$ *is continuous.*

Proof. We prove continuity at each $w \in D_\rho(z_0)$. Let $\epsilon > 0$. By uniform convergence, we can pick $n \in \mathbb{N}$ with $\|P_n - P\|_{\infty, D_\rho(z_0)} < \epsilon/3$. Since polynomials are continuous, there is $\delta > 0$ such that $|P_n(z) - P_n(w)| < \epsilon/3$ for all $z \in D_\delta(w)$. By shrinking δ, we may suppose $D_\delta(w) \subset D_\rho(z_0)$. Then have

$$|P(z) - P(w)| \leqslant |P(z) - P_n(z)| + |P_n(z) - P_n(w)| + |P_n(w) - P(w)| < \epsilon.$$

for all $z \in D_\delta(w)$. $\square$

Remark 5.13. Continuity need not hold on $D(P, z_0)$ (Sierpinski, 1916).

Example 5.14. The series $P = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has $\rho = 1$. On the boundary circle, $|z^n/n^2| \leqslant 1/n^2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and is independent of z. This implies uniform convergence and the proof of Corollary 5.12 shows that P is continuous on its domain $D(P, 0) = \overline{D}_1(0)$.

When the radius of convergence is positive, we drop the word ‘formal’ and simply speak of a **power series**.

Example 5.15. The **exponential series**

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence $+\infty$, since $\frac{z^n}{n!}$ is bounded (even tends to zero), as factorials grow faster than powers. Alternatively, for each $z \in \mathbb{C}$ we have absolute convergence by the ratio test

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Therefore the power series defines a complex function $\exp\colon \mathbb{C} \to \mathbb{C}$. Mathematically, this is a good definition for the exponential function. Using (2.4) we obtain from this the series expansions of the sine and the cosine:

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}. \tag{5.10}$$

Example 5.16 (optional)**.** We first generalize the definition of **binomial coefficients** to complex numbers $\alpha \neq 0$ and $k \in \mathbb{N}$. Set

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{5.11}$$

The **binomial series** is

$$B_\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

For $\alpha \in \mathbb{C} \setminus \mathbb{N}$ we have

$$\left| \frac{\binom{\alpha}{k+1} z^{k+1}}{\binom{\alpha}{k} z^k} \right| = \left| z \frac{\alpha - k}{k+1} \right| \longrightarrow |z| \quad \text{as} \quad k \to \infty.$$

Therefore, the ratio test implies $\rho = 1$ when $\alpha \notin \mathbb{N}$. When $\alpha \in \mathbb{N}$ only finitely many binomial coefficients $\binom{\alpha}{k}$ are non-zero, so B_α is a polynomial and $\rho = +\infty$ and indeed by the binomial theorem we have

$$B_\alpha(z) = (z+1)^\alpha, \qquad \forall \alpha \in \mathbb{N}.$$

Theorem 5.17. *Let P be a formal power series centered at z_0 with radius of convergence $\rho > 0$ and coefficients $(a_n)_{n \in \mathbb{N}}$. Then*

$$P\colon D_\rho(z_0) \longrightarrow \mathbb{C}, \quad z \longmapsto P(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5.12}$$

defines a holomorphic function on $D_\rho(z_0)$ with

$$P'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{5.13}$$

Hence P is infinitely complex differentiable, by induction.

Proof. We show that P is complex differentiable at every $w \in D_\rho(z_0)$. As $\sum_{k=0}^{\infty} kq^k$ converges for $|q| < 1$, the same argument given in (5.9) shows that (5.13) is a convergent series of complex numbers. For simplicity of notation, suppose $z_0 = 0$. We have absolutely convergent series

$$\begin{aligned}
\frac{P(w+h) - P(w)}{h} - P'(w) &= \sum_{n=0}^{\infty} a_n \frac{(w+h)^n - w^n}{h} - P'(w) \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{n} \binom{n}{k} w^{n-k} h^{k-1} \right) - P'(w) \\
&= \sum_{n=0}^{\infty} \sum_{k=2}^{n} a_n \binom{n}{k} w^{n-k} h^{k-1} \\
&= \sum_{k=2}^{\infty} \sum_{n=0}^{\infty} a_n \binom{n}{k} w^{n-k} h^{k-1},
\end{aligned}$$

where we use $\binom{n}{k} = 0$ for $k > n$ and where we have exchanged the order of summation, by absolute convergence. The above is a power series $R(z) = \sum_{k=0}^{\infty} c_k z^k$ with $c_{k-1} = \sum_{n=0}^{\infty} a_n \binom{n}{k} w^{n-k}$ for $k \geqslant 2$ and $c_0 = 0$. Our calculation shows that $R(h)$ converges, so the radius of convergence of R is at least h. Therefore $R(h)$ is continuous on $D_h(z_0)$, so $\lim_{h \to 0} R(h) = R(0) = c_0 = 0$. In other words, $\frac{P(w+h)-P(w)}{h} - P'(w) = R(h) \longrightarrow 0$ as $h \to 0$. □

The **identity theorem** for power series is the following result.

Corollary 5.18. *Let $P = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $Q = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ be a formal power series. If there exists a sequence $w_n \in D(P) \cap D(Q)$ with $\lim_{n\to\infty} w_n = z_0$, $w_n \neq z_0$, and $P(w_n) = Q(w_n)$, then $a_n = b_n$ for all n.*

Proof by induction. Base case $n = 0$. By continuity of P and Q at z_0,

$$\begin{aligned}
a_0 = P(z_0) = P\left(\lim_{n\to\infty} w_n\right) &= \lim_{n\to\infty} P(w_n) \\
= \lim_{n\to\infty} Q(w_n) = Q\left(\lim_{n\to\infty} w_n\right) &= Q(z_0) = b_0.
\end{aligned}$$

Inductive step $n + 1$. Assume that $a_0 = b_0, \dots, a_n = b_n$. By subtracting $\sum_{k=0}^{n} a_k (w_n - z_0)^k = \sum_{k=0}^{n} b_k (w_n - z_0)^k$ from $P(w_n)$ and $Q(w_n)$ we find

$$\sum_{k=n+1}^{\infty} a_k (w_n - z_0)^k = \sum_{k=n+1}^{\infty} b_k (w_n - z_0)^k.$$

As $w_n \neq z_0$, we can divide this equation by $(w_n - z_0)^{n+1}$ and get $\tilde{P}(w_n) = \tilde{Q}(w_n)$ for the formal power series $\tilde{P} = \sum_{k=0}^{\infty} a_{k+n+1}(z - z_0)^k$ and $\tilde{Q} = \sum_{k=0}^{\infty} b_{k+n+1}(z - z_0)^k$. An application of the base case to $\tilde{P}$ and $\tilde{Q}$ shows that $a_{n+1} = b_{n+1}$ for the constant terms, as required. □

Questions for further discussion

- Give precise statements of the comparison test, the ratio test and root test. Recall how the ratio test is proven by comparison with the geometric series.
- Find a power series P with $D(P; 0) = \overline{D}_1(0) \setminus \{\pm 1, \pm i\}$.
- Explain why 'is a bounded sequence' in (5.8) can be replaced by 'is a null sequence'.

Exercises

Exercise 5.1. Let $P = \sum_{n=0}^{\infty} a_n z^n$ and $Q = \sum_{m=0}^{\infty} b_m w^m$ be formal power series with $b_0 = 0$. The **substitution** is the formal power series obtained by expanding

$$P \circ Q = \sum_{\ell=0}^{\infty} c_\ell w^\ell = \sum_{n=0}^{\infty} w^n \left(\sum_{m=0}^{\infty} b_{m+1} w^m \right)^n$$

into powers of w (notice that $b_0 = 0$ implies that each coefficient c_ℓ is a finite sum).

Exercise 5.2. Bernoulli numbers

$$\frac{z}{\exp(z) - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Further resources

- Notebook on uniform convergence

6 Contour integrals

In this section, we will generalize the integral $\int_a^b f(x)dx$ from calculus in two ways. Firstly, we allow $f = u + iv$ to be a complex function. Secondly, we will define the integral over more general curves γ than intervals $[a, b] \subset \mathbb{R}$.

Definition 6.1. Let $f \colon [a, b] \to \mathbb{C}$ be a complex-valued function on an interval with real and imaginary parts $f = u + iv$, $u, v \colon [a, b] \to \mathbb{R}$. Define

$$\int_a^b f(x)dx = \int_a^b u(x)dx + i \int_a^b v(x)dx. \tag{6.1}$$

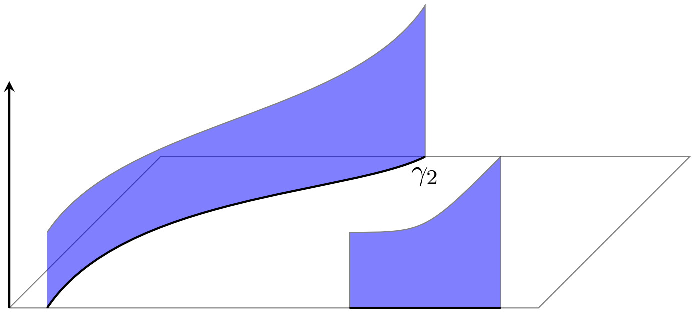

Figure 6.1: Generalizing the integral (the area, up to sign) of a real-valued function over an interval γ_1 to a general differentiable curve γ_2.

Definition 6.2. A **curve** (or **path**) in the plane is a continuous map

$$[a, b] \xrightarrow{\gamma} \mathbb{C}$$

on a closed interval. Decompose $\gamma(t) = u(t) + iv(t)$ into real and imaginary parts. The curve γ is **differentiable** if u, v are differentiable on $[a, b]$ (including one-sided derivatives at the endpoints), and the curve γ is **continuously differentiable** (or C^1) if the derivatives $u'(t), v'(t)$ are continuous on $[a, b]$. The curve γ is **piecewise $\mathbf{C^1}$** if there exists a subdivision of the interval

$$a = t_0 < t_1 < \cdots < t_n = b \tag{6.2}$$

such that each of the restrictions $\gamma|_{[t_{k-1},t_k]}$, $k = 1, \ldots, n$, is a continuously differentiable curve. In this case we call the subdivision **admissible**. For any admissible subdivision, the **length** of the curve γ is

$$L(\gamma) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |\gamma'(t)| dt. \tag{6.3}$$

We call $\gamma([a, b]) \subset \mathbb{C}$ the **image** of the curve. When the image is contained in a subset $D \subset \mathbb{C}$, we say that γ is a **curve in *D***.

A curve is **closed** if $\gamma(a) = \gamma(b)$, and then we call $\gamma(a)$ the **base** of the **loop** (or **contour**) γ.

Definition 6.3. Let $f \colon D \to \mathbb{C}$ be a continuous complex function. Let γ be a piecewise C^1 curve in D. Pick an admissible subdivision (6.2). The **path integral** (or **contour integral** if γ is closed) of f over the curve γ is

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt. \tag{6.4}$$

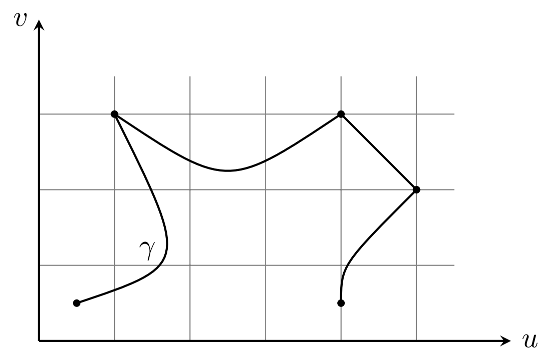

Figure 6.2: A piecewise C^1 curve

Example 6.4. The curve $\gamma(t) = t$ parameterizes the interval $[a, b]$. In this case, (6.4) reduces to the integral $\int_a^b f(x)dx$ from (6.1).

Example 6.5. Let $\gamma(t) = p$ be constant. Then $\int_\gamma f(z)dz = 0$ for all f.

Example 6.6 (important)**.** Let $f(z) = z^n$, $n \in \mathbb{Z}$, and $\gamma(t) = e^{it}$, $t \in [a, b]$.

$$\int_\gamma f(z)dz = \int_a^b e^{int} ie^{it} dt = \begin{cases} i(b-a) & \text{if } n = -1, \\ \frac{e^{i(n+1)b} - e^{i(n+1)a}}{n+1} & \text{if } n \neq -1. \end{cases}$$

In particular, for the **boundary curve** of the disk $\overline{D}_r(z_0)$ defined by

$$[0, 2\pi] \xrightarrow{\gamma_{\partial D_r(z_0)}} \overline{D}_r(z_0), \quad \gamma_{\partial D_r(z_0)}(t) = z_0 + re^{it}. \tag{6.5}$$

In a contour integral, we write this curve simply as $\partial D_r(z_0)$. Then

$$\int_{\partial D_r(z_0)} (z - z_0)^n = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases} \tag{6.6}$$

Lemma 6.7. (a) *The integral* (6.4) *and the length of a curve* (6.3) *are independent of the choice of admissible subdivision.*

(b) *Let* $\varphi\colon [c, d] \to [a, b]$ *be a continuously differentiable function between intervals with* $\varphi(c) = a$ *and* $\varphi(d) = b$*. Let* γ *be a piecewise C^1 curve. Then* $\gamma \circ \varphi$ *is also a piecewise C^1 curve and*

$$\int_{\gamma\circ\varphi} f(z)dz = \int_\gamma f(z)dz. \tag{6.7}$$

Hence the curve integral is independent of the parameterization of γ*.*

Proof. (a) Since every two admissible subdivisions have a common admissible subdivision, it suffices to prove that (6.4) remains unchanged upon passing to a subdivision. Every subdivision is obtained by inductively inserting points, so it suffices to consider the case of a single insertion, say $t_{k-1} < t_* < t_k$. Then only one summand in (6.4) changes, and our claim follows from

$$\int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt = \int_{t_{k-1}}^{t_*} f(\gamma(t))\gamma'(t)dt + \int_{t_*}^{t_k} f(\gamma(t))\gamma'(t)dt.$$

The same argument applies to the length of a curve.

(b) We inductively pick an increasing sequence $s_k \in [c,d]$ with $\varphi(s_k) = t_k$. Take $s_0 = c$. Now suppose that s_k has already been constructed. Then the image of the restricted map $\varphi|_{[s_k,d]}$ contains t_k, b and therefore contains $t_{k+1} \in [t_k, b]$ (intermediate value theorem). Hence there exists $s_{k+1} \in [s_k, d]$ with $\varphi(s_{k+1}) = t_{k+1}$.

By the chain rule, $c = s_0 < s_1 < \cdots < s_n = d$ is an admissible subdivision of $[c,d]$. Using the change of variables $s = \varphi(t)$, we compute

$$\begin{aligned}\int_{\gamma\circ\varphi} f(z)dz &= \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(s))\gamma'(s)ds = \int_\gamma f(z)dz.\end{aligned}$$ □

Due to Lemma 6.7(b) we view curves differing only by a parametrization as **equivalent**. In particular, we may translate and scale the domain $[a,b]$.

Definition 6.8. (a) Let $[a,b] \xrightarrow{\gamma_1} \mathbb{C}$, $[b,c] \xrightarrow{\gamma_2} \mathbb{C}$. Assume $\gamma_1(b) = \gamma_2(b)$. Then the **concatenation** of γ_1 with γ_2 is the curve

$$[a,c] \xrightarrow{\gamma_1 * \gamma_2} \mathbb{C}, \quad (\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a,b], \\ \gamma_2(t) & \text{if } t \in [b,c]. \end{cases} \tag{6.8}$$

(b) The **opposite** of a curve $[a,b] \xrightarrow{\gamma} \mathbb{C}$ is the curve

$$[-b,-a] \xrightarrow{-\gamma} \mathbb{C}, \quad (-\gamma)(t) = \gamma(-t). \tag{6.9}$$

Proposition 6.9. *Let $f, g\colon D \to \mathbb{C}$ be continuous complex functions and let $[a,b] \xrightarrow{\gamma} D$ be a piecewise C^1 curve.*

(a) *The integral over a curve is linear: for all $\lambda \in \mathbb{C}$ we have*

$$\int_\gamma \lambda f(z) + g(z)dz = \lambda \int_\gamma f(z)dz + \int_\gamma g(z)dz. \tag{6.10}$$

(b) *For the opposite curve,*

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz. \tag{6.11}$$

(c) *For the concatenation of curves,*

$$\int_{\gamma_1 * \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz. \tag{6.12}$$

(d) *Suppose that* $|f(z)| \leqslant M$ *for all* $z \in \gamma([a,b])$. *Then*

$$\left| \int_{\gamma} f(z)dz \right| \leqslant M \cdot L(\gamma). \tag{6.13}$$

Proof. (a) follows from the linearity of the ordinary integral and (b) is simply the change of variables $s = -t$ in the integral. The argument for (c) is the same as for Lemma 6.7(a). We prove (d) for a trivial subdivision ($n = 1$):

$$\begin{aligned} \left| \int_{\gamma} f(z)dz \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \\ &\leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\leqslant M \int_a^b |\gamma'(t)|dt. \end{aligned}$$

The general case is obtained by summing this estimate over k. □

Remark 6.10. There is a generalization of the path integral to curves γ of bounded variation called the Riemann–Stieltjes integral. Proposition 6.9 continuous to hold with the length $L(\gamma)$ replaced by the total variation. For example, Lipschitz continuous maps are of bounded variation.

From (6.13) we obtain:

Corollary 6.11. *Let* $(f_n)_{n \in \mathbb{N}}$ *be a uniformly convergent sequence of complex functions on* D. *For every curve* γ *we have*

$$\lim_{n \to \infty} \int_{\gamma} f_n(z)dz = \int_{\gamma} \lim_{n \to \infty} f_n(z)dz. \tag{6.14}$$

Proposition 6.12 (Complex FTC)**.** *Let* f *be a continuous complex function on an open set* U *and let* $[a,b] \xrightarrow{\gamma} U$ *be a piecewise* C^1 *curve. Suppose* F *is a holomorphic function on* U *with* $F' = f$ *(we call* F *a* ***primitive*** *of* f*). Then*

$$\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)). \tag{6.15}$$

In particular, for every closed curve

$$\int_{\gamma} f(z)dz = 0. \tag{6.16}$$

Proof. By the chain rule,

$$\frac{d}{dt}F(\gamma(t)) = f(\gamma(t))\gamma'(t).$$

Using the fundamental theorem of calculus to evaluate (6.4) gives

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} F(\gamma(t_k)) - F(\gamma(t_{k-1})) = F(\gamma(t_n)) - F(\gamma(t_0)). \qquad \square$$

Example 6.13. Since the entire function $f(z) = az + b$ has the primitive $az^2 + bz$, we have $\int_\gamma (az + b)dz = 0$ for every closed curve. More generally, Theorem 5.17 can be used to show that every power series has a primitive.

Remark 6.14. Conversely, every holomorphic function has a holomorphic primitive, provided the domain U is simply-connected (for example, a disk).

Questions for further discussion

- Determine the integral of a power series $P = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $\rho > 0$ along a curve γ in $D_\rho(0)$.

Exercises

Further resources

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Chapter III

Cauchy's theorem and residues

7 Cauchy's theorem

Definition 7.1. Let $D \subset \mathbb{C}$ and $\gamma_0, \gamma_1 \colon [a,b] \to D$ curves in D.

A **homotopy**[1] in D between γ_0, γ_1 is a continuous map

$$\Gamma \colon [0,1] \times [a,b] \longrightarrow D, \quad (s,t) \longmapsto \Gamma_s(t),$$

such that $\Gamma_0(t) = \gamma_0(t)$ and $\Gamma_1(t) = \gamma_1(t)$ for all $t \in [a,b]$. Then γ_0, γ_1 are called (freely) **homotopic**.

If, additionally, $\Gamma_s(a) = p$ and $\Gamma_s(b) = q$ are constant in $s \in [0,1]$, we call Γ a **path homotopy** in D and γ_0, γ_1 **path-homotopic** in D.

A loop is **null-homotopic** in D if it is path homotopic in D to the constant loop.

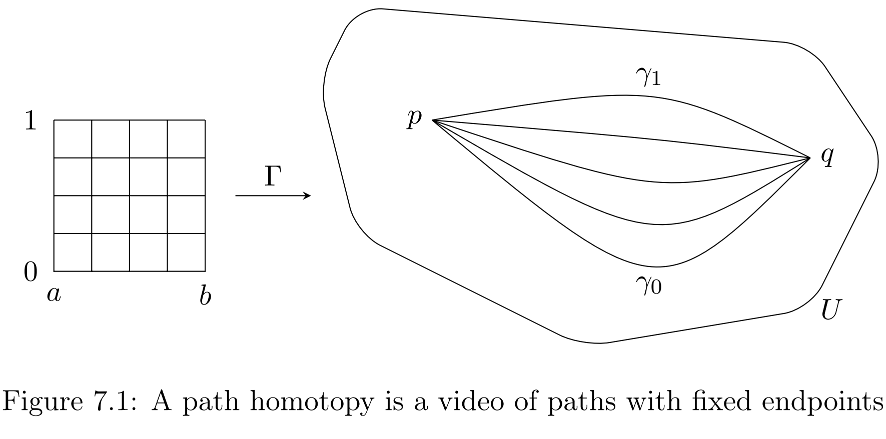

Figure 7.1: A path homotopy is a video of paths with fixed endpoints.

Remark 7.2. In the following we will also suppose that Γ is piecewise C^1, meaning there exist subdivisions

$$0 = s_0 < s_1 < \cdots < s_m = 1, \qquad a = t_0 < t_1 < \cdots < t_n = b$$

[1]From Greek *homos* 'one and the same' and *topos* 'place, region, space'

such that each restriction $\Gamma|_{[s_{j-1},s_j]\times[t_{k-1},t_k]}$ is continuously differentiable. This assumption can be removed.

Example 7.3. A set D is **star-shaped** if there exists a **focal point** $z_0 \in D$ such that for each $z \in D$ the straight line segment $tz + (1-t)z_0$, $t \in [0,1]$, is contained in D. Disks, the complex plane, and rectangles are star-shaped. Every loop γ in a star-shaped domain is null-homotopic. If the focal point agrees with the base of the loop, the path homotopy is

$$\Gamma_s(t) = (1-t)\gamma(t) + tz_0. \tag{7.1}$$

In general, the path homotopy is more complicated to write down. We omit it, since we will not need this fact below.

Figure 7.2: Augustin Louis, Baron Cauchy. Lithograph by J. Boilly, 1821. Wellcome Collection. Public Domain Mark

Theorem 7.4 (Cauchy's theorem)**.** *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open set. Let γ_0, γ_1 be piecewise C^1 curves in U that are path homotopic in U. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz. \tag{7.2}$$

In particular, if γ is path homotopic to the constant loop, then

$$\int_{\gamma} f(z)dz = 0. \tag{7.3}$$

Proof. Pick a null-homotopy $\Gamma\colon [0,1] \times [a,b] \to U$. By reparameterizing, we may assume $[a,b] = [0,1]$. Write $R^{(0)} = [0,1] \times [0,1]$ for the domain of Γ and let $\partial R^{(0)}$ be its piecewise linear boundary path, with the obvious counterclockwise parameterization (see Figure 7.3). The boundary path has

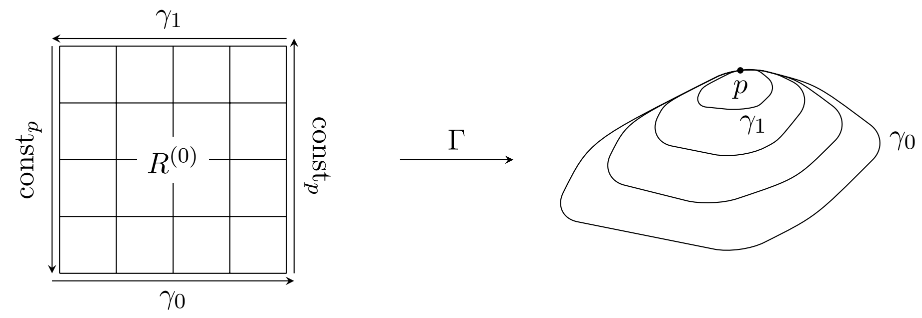

Figure 7.3: Path homotopy of loops.

four segments and the path integral along each constant path vanishes. Using Proposition 6.9(b),(c), we then find that

$$\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\Gamma(\partial R^{(0)})} f(z)dz.$$

Subdivide $R^{(0)}$ into four congruent rectangles $R_i^{(0)}$ as in Figure 7.4. Using the fact that the path integrals in opposite directions cancel, we find

$$\left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| = \left|\sum_{i=1}^{4} \int_{\Gamma(\partial R_i^{(0)})} f(z)dz\right| \leqslant \sum_{i=1}^{4} \left|\int_{\Gamma(\partial R_i^{(0)})} f(z)dz\right|.$$

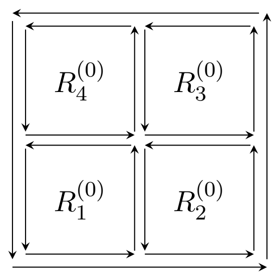

Figure 7.4: Subdivision and cancellation of opposite paths

Let $R^{(1)}$ be the rectangle $R_i^{(0)}$ for which $\left|\int_{\Gamma(\partial R_i^{(0)})} f(z)dz\right|$ is maximal. Then

$$\left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| \leqslant 4 \left|\int_{\Gamma(\partial R^{(1)})} f(z)dz\right|.$$

Now repeat this process with $R^{(1)}$ to obtain $R^{(2)}$ and so forth. This yields a sequence of rectangles $R^{(n)}$ as in Figure 7.5 with sides of length 2^{-n} and

$$\left|\int_{\Gamma(\partial R^{(0)})} f(z)dz\right| \leqslant 4^n \left|\int_{\Gamma(\partial R^{(n)})} f(z)dz\right|. \tag{7.4}$$

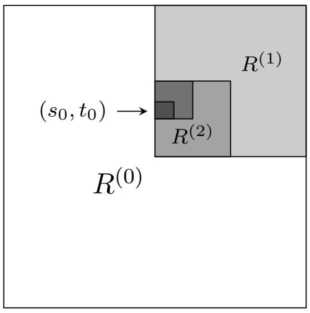

Figure 7.5: Convergent sequence of rectangles $\bigcap R^{(n)} = \{(s_0, t_0)\}$

As the side lengths tend to zero, the midpoints $(s^{(n)}, t^{(n)})$ of the rectangles $R^{(n)}$ are a Cauchy sequence, so converge to some limit (s_0, t_0). Let $z_0 = \Gamma_{s_0}(t_0)$. Since f is complex differentiable at z_0, we may write

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)\rho(z), \qquad \lim_{z\to z_0} \rho(z) = 0. \tag{7.5}$$

By Example 6.13,

$$\int_{\Gamma(\partial R^{(n)})} \big(f(z_0) + (z - z_0)f'(z_0)\big)\, dz = 0. \tag{7.6}$$

As Γ is piecewise C^1, its derivative is bounded in norm by some $C > 0$. Let $\epsilon > 0$. Pick $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and $z \in R^{(n)}$ we have $|\rho(z)| < \epsilon$. As the side lengths of $R^{(n)}$ are 2^{-n} and $z_0 \in R^{(n)}$, we have $|z - z_0| \leqslant \sqrt{2}2^{-n}$ for all $z \in \partial R^{(n)}$. Combining this with (7.5) and (7.6), we can estimate

$$\begin{aligned}\left|\int_{\Gamma(\partial R^{(n)})} f(z)dz\right| &= \left|\int_{\Gamma(\partial R^{(n)})} (z - z_0)\rho(z)dz\right| \\ &\overset{(6.13)}{\leqslant} L(\Gamma(\partial R^{(n)}))\sqrt{2}2^{-n}\epsilon = 4C \cdot 2^{-n}\sqrt{2}2^{-n}\epsilon\end{aligned}$$

Hence

$$\left|\int_\gamma f(z)dz\right| = \left|\int_{R^{(0)}} f(z)dz\right| \overset{(7.4)}{\leqslant} 4C\sqrt{2}\epsilon.$$

As $\epsilon > 0$ is arbitrary, the left hand side must be zero. □

Theorem 7.5. *Let γ_0, γ_1 be piecewise C^1 loops in U that are (freely) homotopic. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz. \tag{7.7}$$

Proof. Let Γ be the homotopy between γ_0 and γ_1. Let $\eta(s) = \Gamma_s(0)$. Figure 7.6 describes the construction of a path homotopy between $\eta * \gamma_0 * (-\eta)$ and γ_1. Hence Theorem 7.4 implies our claim, using the fact that the path

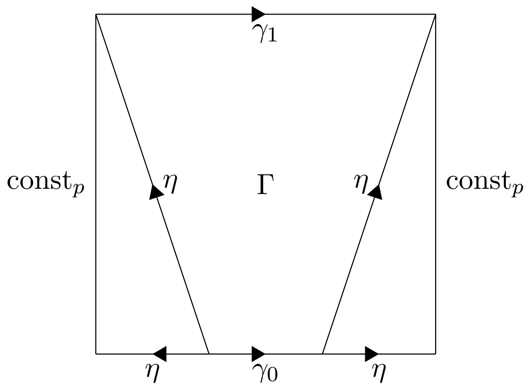

Figure 7.6: Construction of path homotopy

integrals over η and $-\eta$ cancel each other. □

Questions for further discussion

- Why is the map (**??**) continuous?
- How should Figure 7.6 be interpreted? Can you give a formula for the path homotopy (consider cases)?
- Give a counterexample to Cauchy's theorem when (i) γ is not null-homotopic (ii) $f(z)$ is not holomorphic
- What is the analogue of Cauchy's theorem as stated in Theorem **??** in real analysis?
- Does Cauchy's theorem hold for $f(z) = \overline{z}$?

Exercises

Further resources

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8 Applications of Cauchy's theorem

Theorem 8.1 (Cauchy's integral formula)**.** *Let $f: U \to \mathbb{C}$ be a holomorphic function and assume that $\overline{D}_r(z_0) \subset U$ with boundary curve* (6.5)*. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad \forall z \in D_r(z_0). \tag{8.1}$$

Proof. Fix $z \in D_r(z_0)$. Since f is complex differentiable at z, we have

$$f(\zeta) = f(z) + (\zeta - z)f'(z) + (\zeta - z)\rho(\zeta - z), \quad \lim_{\zeta \to z} \rho(\zeta - z) = 0. \tag{8.2}$$

There is a homotopy in $U \setminus \{z\}$ between the loops $\partial D_r(z_0)$ and $\partial D_s(z)$, for all $s > 0$ with $D_s(z) \subset D_r(z_0)$, see Figure 8.1. Moreover, $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ is a

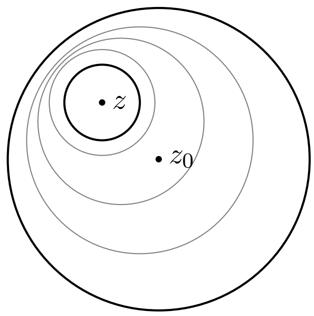

Figure 8.1: Homotopy between $\partial D_r(z_0)$ and $\partial D_s(z)$

holomorphic function on $U \setminus \{z\}$.

$$\begin{aligned}
\int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta &= \int_{\partial D_s(z)} \frac{f(\zeta)}{\zeta - z} d\zeta && \text{by Theorem 7.4} \\
&= \int_{\partial D_s(z)} \frac{f(z)}{\zeta - z} d\zeta && \text{by (8.2)} \\
&\quad + \int_{\partial D_s(z)} f'(z) d\zeta \\
&\quad + \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta \\
&= 2\pi i f(z) && \text{by (6.6) with } n = -1 \\
&\quad + 0 && \text{by (6.6) with } n = 0 \\
&\quad + \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta.
\end{aligned}$$

Since $\lim_{\zeta \to z} \rho(\zeta - z) = 0$ there exists a bound $|\rho(\zeta - z)| \leqslant M$. Hence the final term can be estimated using (6.13) as

$$\left| \int_{\partial D_s(z)} \rho(\zeta - z) d\zeta \right| \leqslant 2\pi s M.$$

This tends to zero as $s \to 0$, and the left hand side is independent of s. $\square$

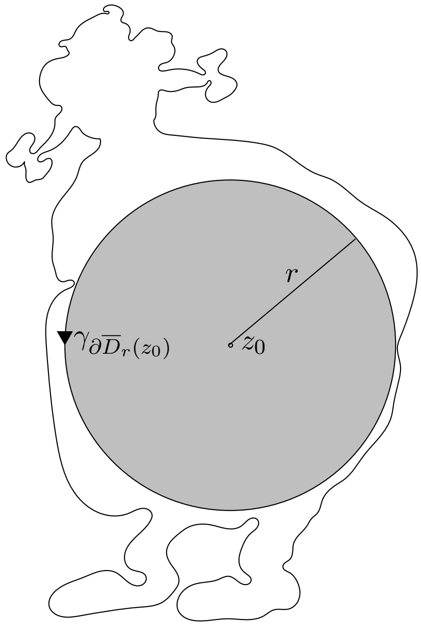

Figure 8.2: Boundary curve of largest disk fitting inside U

Theorem 8.2. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and assume that $\overline{D}_r(z_0) \subset U$. Then f is equal on $D_r(z_0)$ to its Taylor power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{8.3}$$

which has radius of convergence $\rho \geqslant r$. In particular, f is infinitely complex differentiable on the open set U and has a primitive. Moreover,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta. \tag{8.4}$$

Proof. The integrand in (8.1) can be rewritten as

$$\frac{f(\zeta)}{\zeta - z} = \frac{\frac{f(\zeta)}{\zeta - z_0}}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n,$$

where the geometric series converges uniformly for all ζ with $|z - z_0| < |\zeta - z_0|$. In particular, this holds for all $\zeta \in \partial D_r(z_0)$ and using Corollary 6.11 we may exchange the limit and the integral. Hence for all $|z - z_0| < r$ we have a convergent series

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right)(z - z_0)^n. \tag{8.5}$$

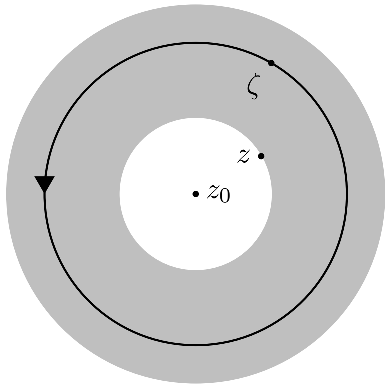

Figure 8.3: The region $|z - z_0| < |\zeta - z_0|$ and the contour of integration

In particular, the series (8.5) has radius of convergence $\rho \geqslant r$.

Finally, (8.4) (and therefore (8.3)) follow from the fact that the power series (8.5) is differentiated termwise, see Theorem 5.17. □

Theorem 8.3 (Liouville). *Every bounded entire function is constant.*

Proof. Suppose $|f(z)| < C$ for all $z \in \mathbb{C}$. For all $z_0 \in \mathbb{C}$ and $r > 0$ we have:

$$\begin{aligned} |f'(z_0)| &= \left| \frac{2}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta \right| && \text{by (8.4) with } n = 2 \\ &\leqslant \frac{2}{2\pi} L(\partial D_r(z_0)) \frac{C}{r^2} = \frac{2C}{r} && \text{by (6.13)} \end{aligned}$$

Letting $r \to \infty$, we find $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Hence f is constant. ***** Prove this earlier! ***** □

Theorem 8.4 (Fundamental theorem of algebra). *Every non-constant polynomial $P(z) = a_n z^n + \ldots + a_1 z + a_0$ with $a_i \in \mathbb{C}$ has a complex root.*

Proof. Assume by contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/P(z)$ is holomorphic and bounded. Indeed, assuming $a_n \neq 0$, we find

$$|P(z)| \geqslant |a_n||z|^n - |a_{n-1}||z|^{n-1} - \ldots - |a_0| \to +\infty \quad \text{as } z \to \infty.$$

Hence $1/|P(z)| \to 0$ as $z \to \infty$. In particular, $|1/P(z)|$ is bounded. □

Definition 8.5. (a) A subset $D \subset \mathbb{C}$ is **path connected** if for all $z_0, z_1 \in D$ there exists a (piecewise C^1) curve γ in D with $\gamma(0) = z_0, \gamma(1) = z_1$.

(b) A path connected subset D is **simply connected** if every loop in D is (freely) homotopic in D to a constant loop.

Example 8.6. Star shaped subsets are simply connected.

Figure 8.4: Carl Friedrich Gauß, 1777-1855, Österreichische Nationalbibliothek. Public Domain

Theorem 8.7. *Let $f\colon U \to \mathbb{C}$ be a holomorphic function on an open set. Suppose U is simply connected and let $z_0 \in U$. Define $F(z)$ for each $z \in U$ by choosing a piecewise C^1 curve γ with $\gamma(0) = z_0$, $\gamma(1) = z$ and defining*

$$F(z) = \int_\gamma f(\zeta)d\zeta. \tag{8.6}$$

Then F is a holomorphic function on U with $F' = f$.

Proof. For any two curves γ_0, γ_1 as in the statement of the theorem, the concatenation $\gamma_0 * (-\gamma_1)$ is a loop which is null-homotopic by assumption. By Theorem 7.5 we have

$$\int_{\gamma_0} f(\zeta)d\zeta - \int_{\gamma_1} f(\zeta)d\zeta = 0,$$

hence (8.6) is well-defined. It remains to check that F is complex differentiable at z, for which it suffices to restrict attention to a disk $\overline{D}_r(z) \subset U$, where Theorem 8.2 implies the existence of a holomorphic primitive g on $D_r(z)$. Hence by the complex FTC (6.15),

$$F(z) = \int_\gamma f(\zeta)d\zeta = g(z) - g(z_0).$$

This equation shows that F is holomorphic with $F' = g' = f$. □

Questions for further discussion

- How should $\int_{-i}^{1+i} z^2 dz$ be interpreted? What is the result?
- Why is $\mathbb{C}^\times$ path connected but not simply connected?
  Hint: consider $\frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz$ for a loop γ in $\mathbb{C}^\times$.
- It is a mysterious calculus fact that $\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ diverges for $|x| > 1$ although $\arctan\colon \mathbb{R} \to \mathbb{R}$ is smooth. In terms of the principal logarithm, $\arctan(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$. Apply this to give a geometric explanation of the divergence using Theorem 8.2.

Exercises

Further resources

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9 Singularities

10 Residue theorem

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