University of Glasgow

School of Mathematics and Statistics

MSc. Project

A Survey of Simplicial Sets

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Gyan Pratap Singh

September 7, 2018

Abstract

A simplicial set is a presheaf over the simplex category $\mathbf{\Delta}$, of finite, non-empty, ordered sets, with weakly order preserving functions between them. This project is targeted towards providing an introduction to simplicial sets and to express their ubiquity with some of their applications. Broadly, the interactions with the categories, **Cat** of small categories and **Top** of topological spaces, have been explored. Particularly, the category $s\mathbf{Set}$ of simplicial sets is the target of right adjoints of the singular and the nerve functors from **Top** and **Cat**, respectively. The former gives a Quillen equivalence between classical model structures, whilst the latter is fully faithful, and realises **Cat** as a reflective subcategory of $s\mathbf{Set}$. Care has been taken to use these functors to generate a bevy of examples in $s\mathbf{Set}$, and to test some of the theory. A number of key results have been highlighted, and an emphasis has been given towards showcasing and motivating the theory by providing examples.

Contents

Introduction

The theory of simplicial sets is central to algebraic topology and higher category theory, and provides a combinatorial approach to topology. This report is a result of compiling material from various sources to serve as an introduction to simplicial sets and all sources referenced are mentioned at the start of each definition and result as they appear in the sources. Unless it is stated explicitly otherwise, it is to be assumed that all proofs have been taken from these references mentioned. The report aims to draw parallels between the combinatorial approach such as that taken in [19], and the categorical approach in [9]. Only material taught up until a standard masters level course in algebraic topology from a standard British university is assumed.

Historical Origin

Simplicial sets were introduced relatively recently by Eilenberg and Zilber in 1950 as “complete semi-simplicial complexes” in [6] for studying singular homology. They were later implemented for studying higher categories, as they are models for $(\infty, 1)$-categories (this will be discussed after Lemma 2.2.11).

Outline

Some of the overall format of this project is loosely based on the structure in [7], [18] and [24].

In Chapter 1, following [17] and [7], the category $s\mathbf{Set}$ of simplicial sets and its generalisation to simplicial objects in an arbitrary category $\mathbf{C}$, was defined with plenty of examples.

Broadly speaking, in Chapter 2 the adjunction

$$h\colon s\mathbf{Set} \rightleftarrows \mathbf{Cat} : N$$

of the homotopy category functor and the nerve functor between the category of simplicial sets and small categories, was considered. This included showing that N is fully faithful, and hence $\mathbf{Cat}$ is equivalent to its image $N(\mathbf{Cat})$, which was then characterised by a combinatorial condition in $s\mathbf{Set}$. During this process, a similar combinatorial condition provided a definition of a generalisation of an ordinary small category, called a **quasi category**.

In Chapter 3, following [18], the category $s\mathbf{Ab}$ of simplicial objects in abelian groups is shown to be equivalent to the category $Ch_{\geq 0}(\mathbf{Ab})$ of non-negatively graded chain complexes on abelian groups, was established. This correspondence is called the **Dold-Kan correspondence**.

In Chapter 4, following [24], the geometric realisation functor $|\cdot|$ from the category $s\mathbf{Set}$ of simplicial sets to the category $\mathbf{Top}$ of topological spaces is constructed as a certain colimit. As a direct consequence of this construction, the geometric realisation functor was seen to be left adjoint to the singular functor $Sing\colon \mathbf{Top} \to s\mathbf{Set}$, used in computing singular homology groups.

In Chapter 5, closely following [19], the homotopy theory exhibited by simplicial sets is explored. The theory is developed purely combinatorially, with analogy to the homotopy theory of topological spaces. A number of familiar results from topology are developed for the category of simplicial sets, including the long exact sequences of simplicial homotopy groups and Hurewicz theorem.

In Chapter 6, the report is concluded by introducing a formal framework for studying categories that exhibit a homotopy theory, called **model categories**, and the notion of equivalences between them is seen in the form of a motivating example of the adjunction

$$|\cdot|\colon s\mathbf{Set} \rightleftarrows \mathbf{Top} : Sing$$

covered in Chapter 4.

1 Simplicial Sets

1.1 Simplicial Sets and Simplicial Identities

Definition 1.1.1 (Simplicial Set; Definition 2.1 in [24])**.** Let $\mathbf{\Delta}$ denote the **simplex category** comprising of objects

$$[n] := \{0, \ldots, n\}$$

for $n \in \mathbb{N}$, and morphisms $f \in \mathbf{\Delta}([n],[m])$, weakly order preserving functions $f\colon [n] \to [m]$ (i.e. $f\colon [n] \to [m]$ for which $f(i) \leq f(j)$ for all $0 \leq i \leq j \leq n$).

- For **Set**, the category of sets, a **simplicial set** is a functor $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ (dually a **cosimplicial set** is a covariant functor $X^\bullet\colon \mathbf{\Delta} \to \mathbf{Set}$). Namely, a simplicial set is a presheaf on the category $\mathbf{\Delta}$.
- Define the category $s\mathbf{Set}$ of simplicial sets to be the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$.

Weakly order preserving maps in $\mathbf{\Delta}$ are composites of maps of the form $d^i\colon [n] \to [n+1]$ called **cofaces**, and maps of the form $s^i\colon [n] \to [n-1]$ called **codegeneracies** where

$$d^i\colon j \mapsto \begin{cases} j, & j < i \\ j+1, & j \geq i \end{cases} \quad \text{and} \quad s^i\colon j \mapsto \begin{cases} j, & j \leq i \\ j-1, & j > i \end{cases}$$

and are determined by relations given in the following lemma.

Lemma 1.1.2 (Cosimplicial Identities; Ch. VII, §5, Lemma 1 in [17])**.** The simplex category $\mathbf{\Delta}$ is generated by the the morphisms d^i and s^i subject to the relations

$$\begin{aligned} d^i d^j &= d^{j+1} d^i \text{ for } i \leq j \\ s^j s^i &= s^i s^{j+1} \text{ for } i \leq j \\ s^j d^i &= \begin{cases} d^i s^{j-1}, & i < j \\ 1, & i = j \text{ or } j+1 \\ d^{i-1} s^j, & i > j+1 \end{cases} \end{aligned}$$

called the **cosimplicial identities**.[1] Moreover every $f \in \mathbf{\Delta}([n],[n'])$ has a unique representation

$$f = d^{i_1} \circ \cdots \circ d^{i_l} \circ s^{j_1} \circ \cdots \circ s^{j_k}$$

with $n' = n - k + l$ and $n' > \cdots > i_1 > \cdots > i_l \geq 0 \leq j_1 < \cdots < j_k < n-1$. Here the empty composite is taken to be the identity map.

Outline of Proof. The cosimplicial identities my be easily verified, and the unique representation follows from them using the fact that the weakly order preserving map $f\colon [n] \to [n']$ is determined by its image $\mathrm{img}(f)$ in $[n']$ and by the non-increasing elements, $k \in [n]$ such that $f(k) = f(k+1)$. The former determines the superscripts of codegeneracies whilst the latter determines that of cofaces. □

Definition 1.1.3 (Simplicial Identities; Ch. VII, §5, Lemma 1 in [17])**.** In view of Lemma 1.1.2, a simplicial set is equivalently a sequence $(X_n := X[n])_{n=0}^{\infty}$, together with **face maps** $d_i := X_\bullet(d^i)$ and **degeneracies** $s_i := X_\bullet(s^i)$ satisfying the relations

$$\begin{aligned} d_i d_{j+1} &= d_j d_i \text{ for } i \leq j \\ s_{j+1} s_i &= s_i s_j \text{ for } i \leq j \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & i < j \\ 1, & i = j \text{ or } j+1 \\ s_j d_{i-1}, & i > j+1 \end{cases} \end{aligned}$$

[1] The notations d^i and s^i do not specify the domains and the codomains. They are made obvious from the context.

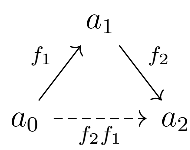
called the **simplicial identities**. Elements of X_n are called n-**simplices** of $X_\bullet$ and in particular, elements of X_0 are called **vertices** of $X_\bullet$. An n-simplex $\sigma \in X_n$ is called **degenerate** if there is an $(n-1)$-simplex $e \in X_{n-1}$ such that $s_i(e) = \sigma$ for some i, and **non-degenerate** if not. In particular, all 0-simplices of $X_\bullet$ are non-degenerate. Abusively, the maps $X^\bullet(d^i)$ and $X^\bullet(s^i)$ will also be called **cofaces** and **codegeneracies**, respectively. The hope is that the distinction is clear from the context.

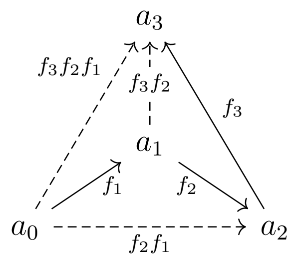
Example 1.1.4 (Simplex Category)**.** The inclusion functor $\mathbf{\Delta} \hookrightarrow \mathbf{Set}$ is a cosimplicial set with the set $[n]$ as its n-simplices and cofaces and codegeneracies s^i and d^i, respectively. Thus it may be regarded as the simplex category $\mathbf{\Delta}$ itself.

Example 1.1.5 ($\mathbf{\Delta}(-,-)$)**.** For $[n] \in \mathrm{obj}(\mathbf{\Delta})$

- the functor $\mathbf{\Delta}([\bullet],[n])\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ represented by $[n]$, is a simplicial set denoted by Δ^n.[2] Its non-degenerate simplices are precisely the injective maps to $[n]$.
- the functor $\mathbf{\Delta}([n],[\bullet])\colon \mathbf{\Delta} \to \mathbf{Set}$ corepresented by $[n]$, is a cosimplicial set denoted by Δ_n.
- the functor $\mathbf{\Delta}([\bullet],[\bullet'])\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \to \mathbf{Set}$ can be regarded as a functor $\mathbf{\Delta}^{\mathrm{op}} \to [\mathbf{\Delta}, \mathbf{Set}]$ and is thus called a simplicial cosimplicial set.[3]

Example 1.1.6 (Nerve of a Category; Example 3.2 in [24])**.** For a small category $\mathbf{C}$ define the **nerve** $N(\mathbf{C})_\bullet$ of $\mathbf{C}$ to be the simplicial set with

- 0-simplices $N(\mathbf{C})_0$, the set of objects $\mathrm{obj}(\mathbf{C})$ of $\mathbf{C}$.
- 1-simplices $N(\mathbf{C})_1$, the set of morphisms $\mathrm{mor}(\mathbf{C})$ of $\mathbf{C}$.
- 2-simplices $N(\mathbf{C})_2$, the set of all pairs of composable morphisms 

of $\mathbf{C}$.
- 3-simplices $N(\mathbf{C})_3$, the set of all triples of composable morphisms 

of $\mathbf{C}$.
- n-simplices $N(\mathbf{C})_n$, n-tuples of compossible morphisms $a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n$ of $\mathbf{C}$ for $n \in \mathbb{N}$.

The face maps $d_i\colon N(\mathbf{C})_n \to N(\mathbf{C})_{n-1}$ for $n \geq 1$ are given by restricting to the face opposite the i^{th} vertex on the commutative n-simplices, more explicitly,

$$d_0\,(a_0 \to \cdots \to a_n) = a_1 \to \cdots \to a_n \quad \text{and} \quad d_n\,(a_0 \to \cdots \to a_n) = a_0 \to \cdots \to a_{n-1}$$

are the restriction onto the last and first $n-1$ arrows respectively, and for $0 < i < n$,

$$d_i\left(a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n\right) = a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} a_{i-1} \xrightarrow{f_{i+1}\circ f_i} a_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} a_n$$

[2]Here the notation Δ^n is used instead of the usual notation $\Delta^n_\bullet$ for brevity. This example is explored in greater detail in Chapter 2.

[3]This terminology will be justified by Definition 5.6.1.

whilst degeneracies $s_i\colon N(\mathbf{C})_n \to N(\mathbf{C})_{n+1}$ given by

$$s_i\left(a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n\right) = a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} a_i \xrightarrow{1} a_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} a_n.$$

(Discussion after Definition 1.1.2.1 in [16]) Notice that the category $\mathbf{C}$ can be recovered up to isomorphism from its nerve $N(\mathbf{C})_\bullet$. By construction, the set of objects and morphisms correspond to the 0-simplices and 1-simplices, respectively. The identity on an object $a \in \mathrm{obj}(\mathbf{C})$ corresponds to the degenerate 1-simplex $s_0\{a\}$, the composition $g \circ f$ of composable morphisms $f, g \in \mathrm{mor}(\mathbf{C})$ corresponds to the edge $d_1(\sigma)$ of a 2-simplex σ with $d_0(\sigma) = g$ and $d_2(\sigma) = f$, with associativity coming from face and degeneracy relations on 3-simplices. Moreover, the property that there is a unique way of composing pairs of composable morphisms in a category, may be translated to a combinatorial condition in s**Set** and used to characterise the image $N(\mathbf{Cat})_\bullet$. This is reserved for a discussion later (in §2.4), and another such characterisation is now briefly remarked.

Remark 1.1.7 (Introduction, pg. 2 in [1])**.** Observe that by construction, there is a pullback

$$\begin{array}{ccc} N(\mathbf{C})_2 & \longrightarrow & N(\mathbf{C})_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ N(\mathbf{C})_1 & \xrightarrow[d_1]{} & N(\mathbf{C})_0 \end{array}$$

in **Set** and more generally, there is a bijection $N(\mathbf{C})_n \cong \overbrace{N(\mathbf{C})_1 \times_{N(\mathbf{C})_0} \cdots \times_{N(\mathbf{C})_0} N(\mathbf{C})_1}^{n \text{ factors}}$. This can be presented as a condition to be satisfied by arbitrary simplicial sets, known as the **Segal condition**, and is satisfied by exactly those simplicial sets that arise as nerves of categories.

1.2 Simplicial Objects

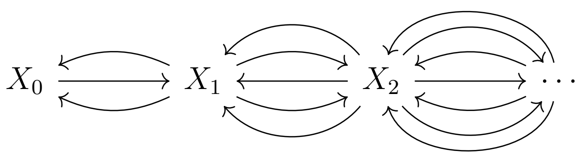
Definition 1.2.1 (Product; Definition 5.1 in [7])**.** Let $X_\bullet$ and $Y_\bullet$ be simplicial sets, define their **product** $(X\times Y)_\bullet$ as the functor $X_\bullet \times Y_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. Namely, for all n, $(X\times Y)_n := (X_n \times Y_n)$ with face maps and degeneracies, $(X\times Y)_\bullet(d^i) = X_\bullet(d^i) \times Y_\bullet(d^i)$ and $(X\times Y)_\bullet(s^i) = X_\bullet(s^i) \times Y_\bullet(s^i)$, the products of face maps and degeneracies of $X_\bullet$ and $Y_\bullet$, respectively.

The product simplicial set $(X \times Y)_\bullet$ is therefore the categorical products of $X_\bullet$ and $Y_\bullet$ in the category of simplicial sets, and comes equipped with a pair of projections onto $X_\bullet$ and onto $Y_\bullet$.

Definition 1.2.2 (Simplicial Object; Definition 2.1 in [24])**.** Generalising Definition 1.1.1, for $\mathbf{C}$ a category, a functor

$$X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$$

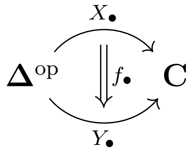
is called a **simplicial object** in $\mathbf{C}$ and dually a functor $X^\bullet\colon \mathbf{\Delta} \to \mathbf{C}$ is called a **cosimplicial object** in $\mathbf{C}$. $s\mathbf{C}$ denotes the category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C}]$ of simplicial objects in $\mathbf{C}$, and objects in $s\mathbf{C}$ are called by the same name as objects in $\mathbf{C}$ but with the added prefix "simplicial" (respectively dually "cosimplicial", for instance "(co)simplicial group" or "(co)simplicial space"). A simplicial object $X_\bullet$ in the category $\mathbf{C}$ is often depicted by the diagram



indicating all face maps going leftwards, and degeneracies going rightwards.[4]

[4]It is non-standard to bend the arrows as it has been done here, however.

In particular, for $X_\bullet, Y_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ simplicial objects in a category $\mathbf{C}$, a **simplicial map** $f_\bullet$ from $X_\bullet$ to $Y_\bullet$, being a morphism in the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C}]$, is a natural transformation

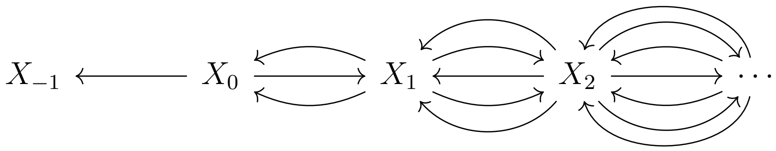


from $X_\bullet$ to $Y_\bullet$. Equivalently, it is a sequence of morphisms $f_i \colon X_i \to Y_i$ in $\mathbf{C}$ rendering diagrams

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{X_\bullet(d^{l_2})} & X_2 & \xrightarrow{X_\bullet(d^{l_1})} & X_1 & \xrightarrow{X_\bullet(d^{l_0})} & X_0 & \xrightarrow{X_\bullet(s^{k_0})} & X_1 & \xrightarrow{X_\bullet(s^{k_1})} & X_2 & \xrightarrow{X_\bullet(s^{k_2})} & \cdots \\
 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\
\cdots & \xrightarrow[Y_\bullet(d^{l_2})]{} & Y_2 & \xrightarrow[Y_\bullet(d^{l_1})]{} & Y_1 & \xrightarrow[Y_\bullet(d^{l_0})]{} & Y_0 & \xrightarrow[Y_\bullet(s^{k_0})]{} & Y_1 & \xrightarrow[Y_\bullet(s^{k_1})]{} & Y_2 & \xrightarrow[Y_\bullet(s^{k_2})]{} & \cdots
\end{array}$$

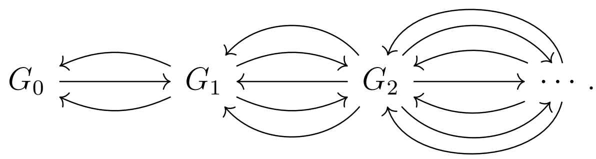
commutative, for all possible indices l_r's and k_s's of face and degeneracy maps. The simplicial map $f_\bullet$ is an **isomorphism** of simplicial objects if it is a natural isomorphism. Equivalently, if in addition to the commutativity of the diagrams above, each f_i is an isomorphism in $\mathbf{C}$.

Definition 1.2.3 (Augmented Simplicial Object; 8.4.6 in [29])**.** Let $\mathbf{\Delta}^+$ denote the **augmented** simplex category with the same objects and morphisms as $\mathbf{\Delta}$, but with the addition of an initial object object $[-1] := \emptyset$. Contravariant functors $X_\bullet \colon \mathbf{\Delta}^+ \to \mathbf{C}$, form special classes of simplicial objects in $\mathbf{C}$ called **augmented** simplicial objects in $\mathbf{C}$, and are depicted by the diagram



where $X_{-1} := X_\bullet([-1])$.

Example 1.2.4 (Simplicial Group; Example 6.2 in [7])**.** Let **Grp** denote the category of groups, a **simplicial group** is a functor $G_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Grp}$, and hence a sequence of groups $(G_n)_{n\in\mathbb{N}}$ together with group homomorphisms d_i and s_i



For μ_{G_n} the multiplication operation on the group G_n, as d_l and s_k are homomorphisms precisely when they are functions such that the squares

$$\begin{array}{ccc}
G_n \times G_n & \xrightarrow{d_l \times d_l} & G_{n-1} \times G_{n-1} \\
{\scriptstyle \mu_{G_n}}\downarrow & & \downarrow{\scriptstyle \mu_{G_{n-1}}} \\
G_n & \xrightarrow[d_l]{} & G_{n-1}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G_n \times G_n & \xrightarrow{s_k \times s_k} & G_{n+1} \times G_{n+1} \\
{\scriptstyle \mu_{G_n}}\downarrow & & \downarrow{\scriptstyle \mu_{G_{n+1}}} \\
G_n & \xrightarrow[s_k]{} & G_{n+1},
\end{array}$$

in **Set** commute, by the definition of face and degeneracies maps on $(G \times G)_\bullet$ from Definition 1.2.1, $G_\bullet$ is equivalently a group object in the category $s\mathbf{Set}$ (with the identity in $G_\bullet$ represented by a map from the terminal object $\mathbf{\Delta}([\bullet], [0])$ in $s\mathbf{Set}$).[5]

[5]On a similar note, it may be shown that simplicial abelian groups are equivalent to monoid objects in simplicial groups using the Eckmann-Hilton argument.

A particular way of generating a simplicial group from a given simplicial set $X_\bullet$ is by applying the functor

$$\begin{aligned} sF\colon s\mathbf{Set} &\to s\mathbf{Grp} \\ X_\bullet &\mapsto FX_\bullet \end{aligned}$$

given by post-composing by the free group functor F, i.e. the functor carrying $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ to the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathbf{Set} \xrightarrow{F} \mathbf{Grp}$.

Remark 1.2.5 (Moore Complex; pg. 5 in [9])**.** Simplicial abelian groups may be generated in a similar way by post-composing with the free abelian group functor $\mathbb{Z}[-]$. From a simplicial abelian group $A_\bullet$, one can form the chain complex $C_\star(A_\bullet)$ with the same objects as $A_\bullet$ at every degree, but with **boundary maps**

$$\partial_n^C := \sum_{i=0}^{n} (-1)^i d_i \colon A_n \to A_{n-1}$$

the alternating sum of face maps, called the **Moore complex** of $A_\bullet$. The assignment $A_\bullet \mapsto C_\star(A_\bullet)$ defines a functor $C_\star$ from simplicial abelian groups to chain complexes $Ch_{\mathbb{Z}}(\mathbf{Ab})$ on abelian groups.[6] For $n \geq 0$, the n^{th} **homology** $H_n(X_\bullet)$ (with integer coefficients) of a simplicial set $X_\bullet$ is the n^{th} homology (with integer coefficients) of the Moore complex of the free abelian group generated by each level, namely the image of $X_\bullet$ under the composite of functors

$$\begin{aligned} s\mathbf{Set} \xrightarrow{s\mathbb{Z}[-]} s\mathbf{Ab} &\xrightarrow{C_\star} Ch_{\mathbb{Z}}(\mathbf{Ab}) \xrightarrow{H_n} \mathbf{Ab} \\ X_\bullet \longmapsto \mathbb{Z}[X]_\bullet &\longmapsto C_\star(\mathbb{Z}[X]_\bullet) \mapsto H_n(C_\star(\mathbb{Z}[X]_\bullet)). \end{aligned}$$

For $X_\bullet^+$ the augmented simplicial set $X_\bullet^+\colon (\mathbf{\Delta}^+)^{\mathrm{op}} \to \mathbf{Set}$ which restricts to $X_\bullet$ on $\mathbf{\Delta}^{\mathrm{op}}$ and maps $[-1]$ to the empty set $\emptyset \in \mathrm{obj}(\mathbf{Set})$, the n^{th} homology group $H_n(X_\bullet^+)$ is defined to be the n^{th} **reduced homology** of $X_\bullet$.

In fact the category of simplicial abelian groups is equivalent to the category of non-negatively graded chain complexes $Ch_{\geq 0}(\mathbf{Ab})$ of abelian groups by the Dold-Kan correspondence. This will be discussed in Chapter 3.

Example 1.2.6 (Topological n-simplices; Ch.1, §1, Example 1.1 in [9])**.** Let **Top** denote the category of topological spaces. There is a cosimplicial space $|\Delta^\bullet|$ defined by

$$\begin{aligned} |\Delta^\bullet|\colon \mathbf{\Delta} &\to \mathbf{Top} \\ [n] &\mapsto |\Delta^n| := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } 0 \leq t_i \right\} \\ ([n] \xrightarrow{d^i} [n+1]) &\mapsto \big(|\Delta^\bullet|(d^i) := B(d^i)\colon (t_0, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n) \big) \\ ([n] \xrightarrow{s^i} [n-1]) &\mapsto \big(|\Delta^\bullet|(s^i) := B(s^i)\colon (t_0, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n) \big) \end{aligned}$$

where each $|\Delta^n|$ is equipped with the subspace topology.[7] Alternatively, the sequence $(|\Delta^n|)_{n\in\mathbb{N}}$ of spaces with the same induced continuous coface and codegeneracy maps as above.

Example 1.2.7 (Singular Complex; Ch.1, §1, Example 1.1 in [9])**.** For W, a topological space, define the **singular complex** or **singular simplicial set** $Sing(W)_\bullet$ of W to be the simplicial set with

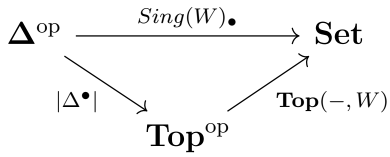
[6] One may check using the simplicial identities, that $\partial_n^C \circ \partial_{n+1}^C = 0$ and so $C_\star(A_\bullet)$ is a chain complex.

[7] The notation $|\Delta^\bullet|$ is logically consistent with the notation $X^\bullet$ for denoting cosimplicial sets and also indicates the dimension of the space. The notation B will be used to denote the composite $|\cdot| \circ N$.

n-simplices $Sing(W)_n$, the set of maps $\mathbf{Top}(|\Delta^n|, W)$. Here $|\Delta^n|$ denotes the geometric n-simplex as defined in Example 1.2.6, together with face and degeneracy maps

$$\begin{aligned} d_i \colon Sing(W)_n &\to Sing(W)_{n-1} \\ f &\mapsto f \circ B(d^i) \end{aligned} \quad \text{and} \quad \begin{aligned} s_i \colon Sing(W)_n &\to Sing(W)_{n+1} \\ f &\mapsto f \circ B(s^i) \end{aligned}$$

given by precomposing with the maps induced by d^i and s^i by the functor $|\Delta^\bullet|$, respectively. In other words $Sing(W)_\bullet$ is the composite of functors expressed by the commutative diagram

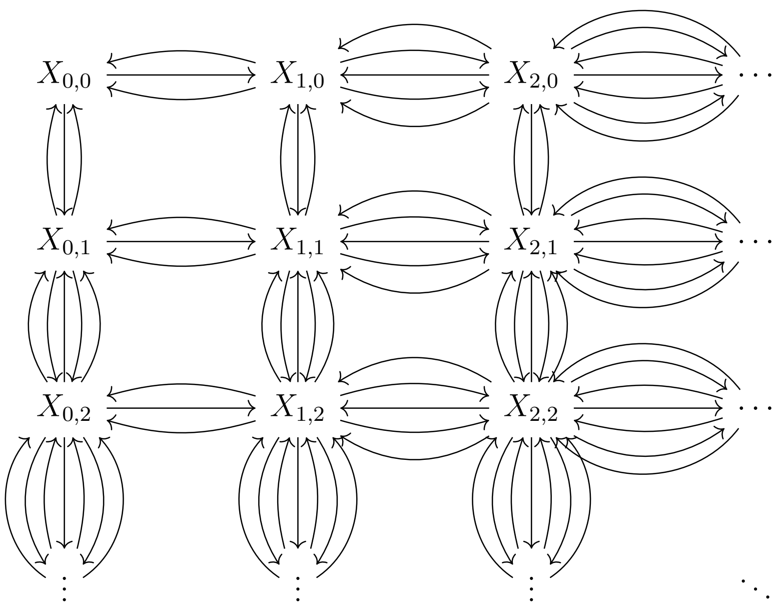


Remark 1.2.8 (Ch.1, §1, Example 1.1 in [9]). The assignment of a topological space to its singular simplicial set defines a functor

$$\begin{aligned} Sing \colon \mathbf{Top} &\to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \\ W &\mapsto Sing(W)_\bullet \\ (V \xrightarrow{f} W) &\mapsto \left(\mathbf{Top}(|\Delta^\bullet|, V) \xrightarrow{f\circ -} \mathbf{Top}(|\Delta^\bullet|, W) \right) \end{aligned}$$

and the Moore complex $C_\star(\mathbb{Z}[Sing(W)]_\bullet)$ (as described in Remark 1.2.5) of the simplicial abelian group $\mathbb{Z}[Sing(W)]_\bullet$ gives the familiar singular chain complex of the space W. Namely, the n^{th} homology of the singular simplicial set $Sing(W)_\bullet$ is the n^{th} singular homology of the space W and considering augmented simplicial sets instead, gives the reduced singular homology of W.

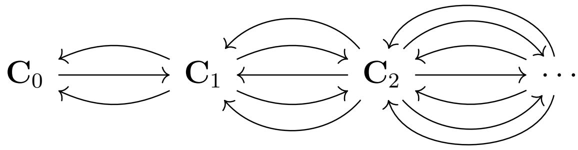
Example 1.2.9 (Bisimplicial Set; Chapter IV, §1, Introduction, pg. 207 in [9]). A simplicial object $(X_{\bullet'})_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to s\mathbf{Set}$ in $s\mathbf{Set}$ is called a **bisimplicial set** and consists of a sequence of simplicial sets $((X_n)_\bullet)_{n\in\mathbb{N}}$ with face and degeneracy maps as maps of simplicial sets, namely each $(X_n)_\bullet$ is a sequence $(X_{n,k})_{k\in\mathbb{N}}$ and consists of face and degeneracy maps $s_{n,i}$ and $d_{n,i}$, respectively, giving rise to the diagram below.



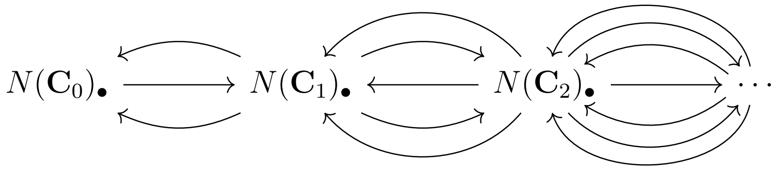
Note, the category $ss\mathbf{Set}$ of bisimplicial sets is isomorphic to the functor category $[\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ as $ss\mathbf{Set} = [\mathbf{\Delta}^{\mathrm{op}}, s\mathbf{Set}] = [\mathbf{\Delta}^{\mathrm{op}}, [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]] \cong [\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$,[8] which is verified by the above diagram.

[8] For **Cat** the category of small categories, this may be seen as similar to the hom-set bijection due to the adjunction $(-) \times \mathbf{\Delta}^{\mathrm{op}} \dashv \mathbf{Cat}(\mathbf{\Delta}^{\mathrm{op}}, -)$. However, this does not hold on the nose as the category **Set** is *not* a small category.

Example 1.2.10 (Simplicial Category; 2.9 in [2]). For **Cat** the category of small categories, a **simplicial category**, $\mathbf{C}_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$ is a sequence of (small) categories $(\mathbf{C}_n)_{n\in\mathbb{N}}$ together with functors s_i and d_i

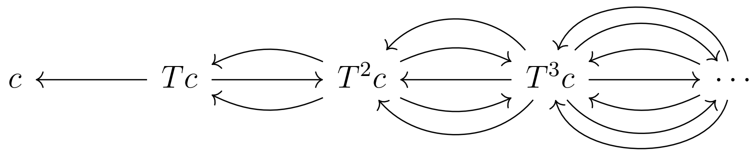


This generalises the usage of the term "simplicial category" used in referring to categories enriched over simplicial sets. In particular, the special case when $\mathrm{obj}(\mathbf{C}_0) = \mathrm{obj}(\mathbf{C}_1) = \mathrm{obj}(\mathbf{C}_2) = \cdots$ and s_i and d_i are all identities on objects, defines a category whose objects are $\mathrm{obj}(\mathbf{C}_i)$ and morphisms whose from c to c' form the simplicial set $\mathbf{C}_\bullet(c, c')$. Conversely, it is easy to see that any simplicially enriched category arises this way. Moreover, a simplicial object in the category of small categories may be viewed as a special case of bisimplicial set as follows: since the nerve of a category is a simplicial set (see Example 1.1.6), the nerve applied to each level,



of the simplicial category $\mathbf{C}_\bullet$, namely the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathbf{C}_\bullet} \mathbf{Cat} \xrightarrow{N} s\mathbf{Set}$, is a bisimplicial set.[9]

Example 1.2.11 (Comonad; Introduction, pg. 2 in [26] and 1.9, Introduction, pg. 4 in [8]). A comonad $(T \colon \mathbf{C} \to \mathbf{C}, \delta \colon T \implies T^2, \epsilon \colon T \implies 1_{\mathbf{C}})$ on a category $\mathbf{C}$, such as one coming from an adjunction, gives rise to an (augmented) simplicial object



in $\mathbf{C}$, for each $c \in \mathrm{obj}(\mathbf{C})$,[10] with face maps

$$d_i := T^i \epsilon_{T^{n-i}c} \colon T^{n+1}c \to T^n c \quad \text{and degeneracies} \quad s_i := T^i \delta_{T^{n-i}c} \colon T^{n+1}c \to T^{n+2}c$$

Let $\vec{\mathbf{Grph}}$ denote the category of directed graphs, then in the special case of the comonad $(T = FU, \delta = F\eta U, \epsilon)$ from the free-forgetful adjunction

$$F \colon \vec{\mathbf{Grph}} \rightleftarrows \mathbf{Cat} : U$$

between directed graphs and small categories, where η is the unit and ϵ is the counit, the corresponding simplicial object in **Cat** for a fixed category, has the same objects at each level and face maps and degeneracies acting as identities on objects and thus defines, just as it did in the case of Example 1.2.10, a category enriched over simplicial sets.

[9] This will be fully justified once it has been shown that the functor N forms a fully faithful functor.

[10] Dually a monad gives rise to an (augmented) cosimplicial object in $\mathbf{C}$.

2 Quasi Categories and their Homotopy Categories

From this point on, set $\mathbf{\Delta}$ to be the full subcategory of $\mathbf{Cat}$, with objects given by the sequence $(\llbracket n \rrbracket)_{n=0}^{\infty}$, where $\llbracket n \rrbracket$ is the category $0 \to \cdots \to n$ with $n+1$ distinct objects and exactly n non-identity morphisms between them, as shown schematically. Defined this way, $\mathbf{\Delta}$ is isomorphic to the category $\mathbf{\Delta}$ from the previous chapter with $[n] \leftrightarrow \llbracket n \rrbracket$, hence justifying the notational abuse. Simplicial sets and face and degeneracy maps are analogously defined.

2.1 Nerves of Categories

Definition 2.1.1 (Nerve; Example 3.2 in [24])**.** Formalising Example 1.1.6, as $\mathbf{\Delta} \subseteq \mathbf{Cat}$ there is a well defined functor

$$\begin{aligned} N\colon \mathbf{Cat} &\to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] \\ \mathbf{C} &\mapsto N(\mathbf{C})_\bullet := \mathbf{Cat}(\llbracket \bullet \rrbracket, \mathbf{C}) \\ (\mathbf{C} \xrightarrow{F} \mathbf{D}) &\mapsto \Big(\mathbf{Cat}(\llbracket \bullet \rrbracket, \mathbf{C}) \xrightarrow{F\circ -} \mathbf{Cat}(\llbracket \bullet \rrbracket, \mathbf{D})\Big) \end{aligned}$$

called the **nerve functor** and the image $N(\mathbf{C})_\bullet$ under N of a small category $\mathbf{C}$ is called the **nerve** of the category $\mathbf{C}$.

The nerve functor restricts to the Yoneda embedding $\mathbf{\Delta} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ sending $\llbracket n \rrbracket$ to the functor Δ^n represented by $\llbracket n \rrbracket$ (see Example 1.1.5).

Example 2.1.2 ($\mathbf{B}G_\bullet$; Example 2.7 in [28])**.** Let G be a group, then consider the category $\mathbf{B}G$ associated (functorially) to G, consisting of a single object $\mathrm{obj}(\mathbf{B}G) = \{\star\}$ and a morphism $\star \circlearrowleft_{g \in G}$ for each group element, under compositions defined by multiplication in the group G. A functor between two such groupoids $\mathbf{B}G$ and $\mathbf{B}H$ associated to groups G and H, is equivalent to giving a group homomorphism between G and H, and so the assignment $G \mapsto \mathbf{B}G$ defines a fully faithful functor $\mathbf{B}\colon \mathbf{Grp} \to \mathbf{Cat}$. The nerve $N(\mathbf{B}G)_\bullet$ (denoted $\mathbf{B}G_\bullet$, for brevity), of the category $\mathbf{B}G$ has

- a 0-simplex, the unique functor $\llbracket 0 \rrbracket \to \mathbf{B}G$. Thus $\mathbf{B}G_0$ is a singleton.
- 1-simplices, the set of functors $\llbracket 1 \rrbracket \to \mathbf{B}G$. Thus there is a bijection $\mathbf{B}G_1 \cong G$.
- 2-simplices, the set of functors $\llbracket 2 \rrbracket \to \mathbf{B}G$. Since for each pair of elements $(g, h) \in G \times G$, there is a commutative triangle $\begin{array}{ccc} \star & \xrightarrow{g} & \star \\ & \overset{hg}{\searrow} & \downarrow h \\ & & \star \end{array}$, there is a bijection $\mathbf{B}G_2 \cong G \times G$.
- n-simplices, the set of functors $\llbracket n \rrbracket \to \mathbf{B}G$. Since for each n-tuple of elements $(g_1, \dots, g_n) \in G^{\times n}$, there is a commutative n-simplex, there is a bijection $\mathbf{B}G_n \cong G^{\times n}$.

The faces, being pre-composition by d^i, correspond to

$$\begin{aligned} d_i\colon \mathbf{B}G_n &\to \mathbf{B}G_{n-1} \\ (g_1, \dots, g_n) &\mapsto (g_1, \dots, g_{i-2}, g_i g_{i+1}, g_{i+2}, \dots, g_n) \end{aligned}$$

for $0 < i < n$ and restrictions to first and last $(n-1)$-tuples for d_n and d_0, respectively. Here the 0-tuple is taken to be the point $\star$ in $\mathbf{B}G$. The degeneracies are given by

$$\begin{aligned} s_i\colon \mathbf{B}G_n &\to \mathbf{B}G_{n+1} \\ (g_1, \dots, g_n) &\mapsto (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n) \end{aligned}$$

where e is the neutral element in the group.[11] This also explains the connection to Example 1.1.6.

[11] Thus if G is abelian, faces and degeneracies are group homomorphisms, and $\mathbf{B}G_\bullet$ is a simplicial abelian group.

Definition 2.1.3 (n-simplex; Example 1.4 in [18]). As another example, the nerve of the category $[\![n]\!]$ is called the (simplicial) n-**simplex**, namely the simplicial set

$$\begin{aligned} \Delta^n \colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathbf{Set} \\ [\![k]\!] &\mapsto \Delta^n_k := \mathbf{\Delta}([\![k]\!], [\![n]\!]) \\ ([\![k-1]\!] \xrightarrow{d^i} [\![k]\!]) &\mapsto \Big(\mathbf{\Delta}([\![k]\!], [\![n]\!]) \xrightarrow{-\circ d^i} \mathbf{\Delta}([\![k-1]\!], [\![n]\!]) \Big) \\ ([\![k+1]\!] \xrightarrow{s^i} [\![k]\!]) &\mapsto \Big(\mathbf{\Delta}([\![k]\!], [\![n]\!]) \xrightarrow{-\circ s^i} \mathbf{\Delta}([\![k+1]\!], [\![n]\!]) \Big) \end{aligned}$$

Since by Lemma 1.1.2, every morphism in Δ^n_k may be obtained by post-composing the identity $1_{[\![n]\!]} \colon [\![n]\!] \to [\![n]\!]$ with codegeneracies followed by cofaces, Δ^n is equivalently the simplicial set generated by taking all faces and degeneracies of the identity functor $1_{[\![n]\!]}$.

The motivation for the definition is provided by the following universal property.

Lemma 2.1.4 (Proposition 1.7 in [18]). Let $X_\bullet$ be a simplicial set, for each n there is a natural bijection

$$X_n \cong s\mathbf{Set}(\Delta^n, X_\bullet)$$

Proof. As $s\mathbf{Set}(\Delta^n, X_\bullet) = [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta^n, X_\bullet)$, this is a special case of Yoneda's lemma (see Ch. III, §2 in [17]). □

With this noted maps $\Delta^n \to X_\bullet$ are sometimes called n-**simplices** of $X_\bullet$. Moreover the following theorem holds

Theorem 2.1.5 (Density Theorem; Ch. III, §7, Theorem 1 in [17]). There is an isomorphism

$$X_\bullet \cong \underset{x \in X_n}{colim}\,(\Delta^n)$$

where the index $x \in X_n$ is treated as a simplicial map $\Delta^n \to X_\bullet$ justified by Lemma 2.1.4.

2.2 Kan Conditions

Definition 2.2.1 (Subsimplicial Set; Ch. 5, Introduction, pg. 5 in [24]). A **subsimplicial set** $A_\bullet$ of a simplicial set $X_\bullet$ is a simplicial set such that for each n, $A_n \subseteq X_n$ and the face and degeneracy maps for $A_\bullet$ are restrictions of those for $X_\bullet$.

Definition 2.2.2 (Horn; Definitions 5.2 and 5.7 in [24]). For $n \geq 1$ and $0 \leq k \leq n$, the $(n,k)^{th}$ **horn** Λ^n_k is the simplicial set obtained from from Δ^n, by removing the n-simplex and the face opposite its k^{th} vertex. More formally, the subsimplicial set of Δ^n generated by taking all face maps and degeneracies of the set of $(n-1)$-simplices

$$\big\{ d^i \colon [\![n-1]\!] \to [\![n]\!] \;\big|\; i \in \{0, \dots, k-1, k+1, \dots, n\} \big\}$$

in Δ^n_{n-1}. By Lemma 1.1.2, this means that $(\Lambda^n_k)_m$ consists of all maps $f \colon [\![m]\!] \to [\![n]\!]$ which can be factored as $f = d^i \circ f'$ where $f' \in \Delta^{n-1}_m$ and $i \neq k$.

If $k = 0$ or n, Λ^n_k is called an **outer horn**, and an **inner horn** if otherwise. An inner (respectively outer) **horn in** $X_\bullet$ is a simplicial map $\Lambda^n_k \to X_\bullet$ where Λ^n_k is an inner (respectively outer) horn.

Remark 2.2.3 (Example 5.2 in [24]). Let Λ^n_k be a horn,[12] then

- for $m < n-1$, $(\Lambda^n_k)_m = \Delta^n_m$,

[12] For brevity, the notation Λ^n_k lacks a bullet in the subscript. The symbol Λ is suggestive of the case when $n = 2$.

- $(\Lambda^n_k)_{n-1}$ consists of all functors in Δ^n_{n-1} whose unique representations provided by Lemma 1.1.2 do not contain d^k, and
- for $m \geq n$ any functor in $(\Lambda^n_k)_m$ contains codegeneracies in its representation provided by Lemma 1.1.2, i.e. all m-simplices are degenerate.

Definition 2.2.4 (Kan Condition; Definition 5.5 in [24])**.** A simplicial set $X_\bullet$ satisfies the **Kan condition** if every horn in $X_\bullet$ has a **filler**, that is to say, if for all horns Λ^n_k, and diagrams of solid arrows of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X_\bullet \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

there is an extension given by the dashed arrow, such that the resulting triangle commutes. In this situation, $X_\bullet$ is called a **Kan complex**. If instead the simplicial set is only required to satisfy that all inner horns have fillers, then it is called a **quasi category**. **Kan** is used to denote the full subcategory of s**Set** consisting of Kan complexes.

Remark 2.2.5 (Combinatorial Kan Condition; Definition 1.3 in [19])**.** The Kan condition is equivalent to the condition that for every n-tuple of $(n-1)$-simplices $(x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n)$ in $X_\bullet$,[13] that are **compatible**, that is, satisfying the $d_i x_j = d_{j-1} x_i$ for all $i, j \neq k$ with $i < j$, there is an n-simplex $x \in X_n$ satisfying $d_l x = x_l$ for each $l \neq k$, similarly for quasi categories, only $0 < k < n$ is required to be satisfied.[14] The map $\Lambda^n_k \to X_\bullet$ corresponding to the tuple $(x_0, \ldots, x_{k-1}, -, x_{k+1}, \ldots, x_n)$ is written

$$\Lambda^n_k \xrightarrow{(x_0,\ldots,x_{k-1},-,x_{k+1},\ldots,x_n)} X_\bullet.$$

Example 2.2.6 ($\Delta^1 \notin \mathrm{obj}(\mathbf{Kan})$; Example 7.4 in [7])**.** Δ^1 is *not* a Kan complex as the horn

$$\Lambda^2_0 \xrightarrow{(-,\, d^0 s^0,\, 1_{[\![1]\!]})} \Delta^1$$

which may be formed as the compatible condition is satisfied by the cosimplicial identities, does not have a filler. Suppose there was a 2-simplex $x \in \Delta^1_2$ satisfying $d_1(x) = 1_{[\![1]\!]}$ and $d_2(x) = d^0 s^0$, then $0 = 1_{[\![1]\!]}(0) = d_1(x)(0) = x \circ d^1(0) = x(0) = x \circ d^2(0) = d_2(x)(0) = d^0 s^0(0) = d^0(0) = 1$, a contradiction. It will be made clear however, that Δ^1 is a quasi category.

Example 2.2.7 ($Sing(\mathbf{Top}) \subseteq \mathbf{Kan}$; Example 7.6 in [7])**.** The singular simplicial set $Sing(W)_\bullet$ of a space W, as seen in Example 1.2.7, satisfies the Kan condition as given $k \leq n$, the dashed arrow

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Sing(W)_\bullet \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

making the resulting diagram commute, exists when the map

$$s\mathbf{Set}(\Delta^n, Sing(W)_\bullet) \to s\mathbf{Set}(\Lambda^n_k, Sing(W)_\bullet)$$

[13] The term "n-tuple" is somewhat of an abuse. It is really an $(n+1)$-tuple with the k^{th} index omitted.

[14] Having a combinatorial point of view as remarked in Remark 2.2.5 and a categorical point of view provided by the extension problem stated in the Definition 2.2.4 preceding it, will be a recurring theme in consequent sections. Sometimes one point of view is more favourable than the other, for instance the latter is, in Example 2.2.8.

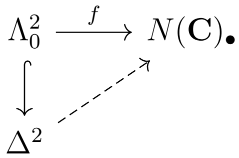
obtained by precomposing with the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, is a surjection. However, by the adjunction $|\cdot| \dashv Sing$,[15] this is equivalent to requiring the corresponding map

$$Sing(W)_n = \mathbf{Top}(|\Delta^n|, W) \to \mathbf{Top}(|\Lambda^n_k|, W)$$

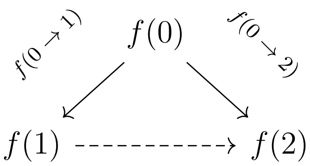
to be surjective, where $|\Lambda^n_k| := \partial|\Delta^n| \backslash \{(t_0, \ldots, t_n) \in |\Delta^n| : t_k = 0\}$.[16] Which is true as its right inverse is given by precomposing with the retraction $|\Delta^n| \to |\Lambda^n_k|$ in **Top**.

Example 2.2.8 ($s\mathbf{Grp} \subseteq \mathbf{Kan}$; Theorem 17.1 in [19])**.** A simplicial group $G_\bullet$, viewed as a special case of a simplicial set, is a Kan complex. There is an explicit algorithm for computing the filler of a given horn in $G_\bullet$ (see proof of Theorem 17.1 of [19]). This was a result by Moore in 1954.

Example 2.2.9 ($N(\mathbf{Cat}) \nsubseteq \mathbf{Kan}$; Remark 1.1.2.3 in [16])**.** The nerve $N(\mathbf{C})_\bullet$ of a small category $\mathbf{C}$ is not always a Kan complex as the Kan condition



is a restatement of the lifting problem for the existence of a dashed arrow for given solid arrows

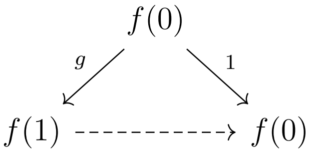


in $\mathbf{C}$, which does not always admit a solution.

Notice that the lifting problem above *does* admit a solution when $\mathbf{C}$ is a **groupoid** (that is if all morphisms in $\mathbf{C}$ are invertible). This is indeed also a necesary condition for $N(\mathbf{C})_\bullet$ to be a Kan complex.

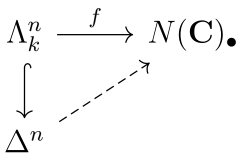
Proposition 2.2.10 (Ch. 1, §3, Lemma 3.5 in [9])**.** For $\mathbf{C}$ a small category $N(\mathbf{C})_\bullet$ is a Kan complex if and only if $\mathbf{C}$ is a groupoid.

Sketch of Proof. ($\implies$) If $N(\mathbf{C})_\bullet$ is a Kan complex, then for $g \in \mathrm{mor}(\mathbf{C})$ the special case



of the lifting problem from Example 2.2.9 admits a solution $g^{-1} \in \mathrm{mor}(\mathbf{C})$. Thus $\mathbf{C}$ is a groupoid.

($\impliedby$) Conversely, if $\mathbf{C}$ is a groupoid, then for a given horn and solid arrows



the dashed map $\Delta^n \to N(\mathbf{C})_\bullet$ is obtained by filling in the (missing) face opposite the k^{th} vertex by constructing morphisms $f(i) \to f(j)$ where $k \neq i$ or j as composites $f(i) \to f(k) \to f(j)$ by reversing the direction of either arrow if required. Seek the refernce for an in depth proof. □

[15] This awaits justification and is the statement of Lemma 4.4.1 which is yet to be proved.

[16] This will be shown in Example 4.3.4.

In the special cases of Examples 2.2.7 and 2.2.8, Proposition 2.2.10 implies in particular that the nerve $N(\mathbf{C})_\bullet$ of a (small) category $\mathbf{C}$ can not have the structure of a simplicial group or a singular complex unless $\mathbf{C}$ is a groupoid. However, a weakening of the Kan condition holds for $N(\mathbf{C})_\bullet$.

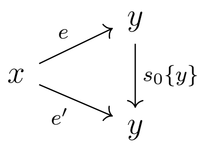
Lemma 2.2.11 (Example 1.1.2.6 in [16])**.** For $\mathbf{C}$ a small category $N(\mathbf{C})_\bullet$ is a quasi category.

Proof. This will follow immediately as a special case, once the image of the nerve functor has been characterised (Theorem 2.4.1). □

As N is fully faithful,[17] **Cat** is equivalent to its image $N(\mathbf{Cat})$ and thus, as the terminology suggests, a quasi category is viewed as a generalisation of a category, where compositions are not necessarily unique. From this prospective, the inner horn filling condition for quasi categories may be interpreted as the requirement that for all $k > 1$, k-morphisms are required to have inverses. For this reason a quasi category is sometimes called an **∞-category** or an **$(\infty, 1)$-category** (see Examples 2.3.6 and 5.2.5 for an instance of a similar characterisation for Kan complexes).[18]

2.3 Homotopy Category of a Simplicial Set

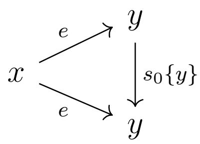
Definition 2.3.1 (Homotopy Between Edges; Definition 3.1 in [19])**.** For $X_\bullet$ a simplicial set, two 1-simplices $x \underset{e'}{\overset{e}{\rightrightarrows}} y$ in X_1 are called **homotopic** (written $e \overset{1}{\sim} e'$)[19] if there is a 2-simplex $\sigma \in X_2$ with boundary



i.e. if $d_0(\sigma) = s_0\{y\}$, $d_1(\sigma) = e'$ and $d_2(\sigma) = e$.

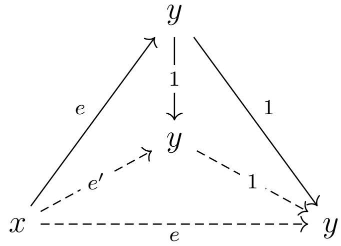
Proposition 2.3.2 (Proposition 3.2 in [19])**.** If $X_\bullet$ is a quasi category, then the homotopy relation $\overset{1}{\sim}$ is an equivalence relation on X_1.

Proof. (One dimensional case of the proof of Proposition 3.2 in [19]) (Reflexivity) Suppose $x \xrightarrow{e} y$ is a given 1-simplex then $e \overset{1}{\sim} e$ as



is the boundary of the degenerate 2-simplex $s_1(e)$.

(Symmetry) Suppose $e \overset{1}{\sim} e'$, then there is a horn



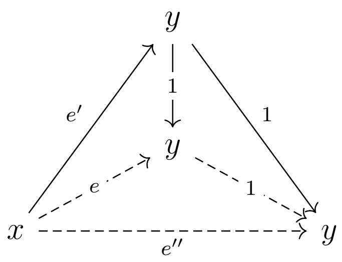
[17]This is yet to be shown (in Example 2.3.5).

[18]Sometimes the term $(\infty, 1)$-category refers to a more general class of categories with some notion of n-morphisms for each n, and k-morphisms equivalences for each $k \geq 1$.

[19]For a Kan complex $X_\bullet$, the notation $\overset{1}{\sim}$ for this equivalence relation on X_1 is a special case when $n = 1$, for the notation $\overset{n}{\sim}$ on X_n to be defined in Definition 5.2.1.

missing its bottom face, where 1 denotes the degenerate 1-simplex $s_0\{y\}$ on y, where its other two filled 2-faces are given by $e \overset{1}{\sim} e'$ and $e \overset{1}{\sim} e$. By the inner horn filling condition, there is a filler rendering the diagram depicted, a boundary of a (degenerate) 3-simplex. In particular, the filled bottom face gives by definition $e' \overset{1}{\sim} e$.

(Transitivity) Suppose $e \overset{1}{\sim} e'$ and $e' \overset{1}{\sim} e''$, then by symmetry, $e' \overset{1}{\sim} e$ and so the same argument applies to the horn



missing its bottom face, to show that $e \overset{1}{\sim} e''$. □

Lemma 2.3.3 (Discussion on pg. 13 in [16] and 2.3.4 in [12])**.** For $x, y \in X_0$, define the set $X_1(x, y) := \{e \in X_1 \mid d_1(e) = x \text{ and } d_0(e) = y\}$. For $X_\bullet$ a quasi category and $f, g \in X_1$ such that $d_0(f) = d_1(g)$ (i.e. that are compatible), let α be a filler of the horn

$$\Lambda^2_1 \xrightarrow{(g,-,f)} X_\bullet$$

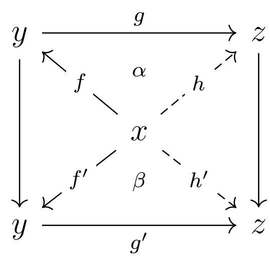
in $X_\bullet$ and $h := d_1(\alpha)$. The assignment $([g], [f]) \mapsto [h]$ where the square brackets denote homotopy classes, defines a well defined map

$$\left(X_1(y, z)/\overset{1}{\sim}\right) \times \left(X_1(x, y)/\overset{1}{\sim}\right) \to X_1(x, z)/\overset{1}{\sim}.$$

Proof. Suppose $f \overset{1}{\sim} f'$ and $g \overset{1}{\sim} g'$ and that there are fillers α and β of the horns

$$\Lambda^2_1 \xrightarrow{(g,-,f)} X_\bullet \quad \text{and} \quad \Lambda^2_1 \xrightarrow{(g',-,f')} X_\bullet$$

in $X_\bullet$, respectively. Set $h := d_1(\alpha)$ and $h' := d_1(\beta)$ then there are simplices depicted by



in $X_\bullet$, where the two vertical arrows are degenerate 1-simplices, and the two horizontal arrows have the same source, y. It is clear from the triangle formed on the right that $h \overset{1}{\sim} h'$. □

Definition 2.3.4 (Homotopy Category; 2.3.4 in [12])**.** In fact, for $X_\bullet$ a quasi category, there is a category $Ho(X_\bullet)$ called its **homotopy category**, with the set of vertices X_0 as objects and homotopy classes of edges $Ho(X_\bullet)(x, y) := X_1(x, y)/\overset{1}{\sim}$ as morphisms. The identity class $[1_x] \in Ho(X_\bullet)(x, x)$ corresponding to an object in $x \in X_0$ is given by the homotopy class $[s_0(x)]$ of the degenerate 1-simplex on x, and associativity comes from relations on 3-simplices.

Example 2.3.5 ($N(\mathbf{C})_\bullet$; 4.5.14 (vi) in [25]). Since degenerate 1-simplices in the nerve $N(\mathbf{C})_\bullet$ of a small category $\mathbf{C}$ precisely correspond to identity maps, two simplices in $N(\mathbf{C})_1$ are homotopic if and only if they are equal, thus the homotopy category $Ho(N(\mathbf{C})_\bullet)$ of $N(\mathbf{C})_\bullet$ is isomorphic to $\mathbf{C}$. This shows in particular that N must be fully faithful.

Example 2.3.6 (Fundamental Groupoid; Example 1.1.2.5 in [16]). For W a topological space, the homotopy category

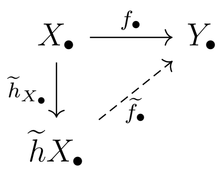
$$\pi_{\leq 1}(W) := Ho(Sing(W)_\bullet)$$

of the singular simplicial set $Sing(W)_\bullet = \mathbf{Top}(|\Delta^\bullet|, W)$, which is a Kan complex (hence a quasi category) from Example 2.2.7, is called the **fundamental groupoid** of the space W and consists of points of W as objects and homotopy classes of paths between pairs of points as morphisms.[20]

Once homotopies between simplices have been defined, more generally, the **fundamental n-groupoid** $\pi_{\leq n}(W)$ will be defined (in Example 5.2.5), with the fundamental groupoid being the special case when $n = 1$.

Remark 2.3.7. Note that there is an inherent claim in Example 2.3.6, that for a topological space W, a homotopy between 1-simplices of the singular complex $Sing(W)$, are path homotopies between in the space W. Skip to Example 5.2.3 for the justification of this fact.

Definition 2.3.8 (Homotopy Category). As it is defined so far, the homotopy category is only well defined for quasi categories. However, a functor $s\mathbf{Set} \to \mathbf{Cat}$ may be constructed which restricts to Ho. As $\mathbf{Cat}$ has all small limits, and N is fully faithful, so does its image $N(\mathbf{Cat})$. Moreover, as a full subcategory of $s\mathbf{Set}$, the forgetful functor $N(\mathbf{Cat}) \hookrightarrow s\mathbf{Set}$ has a left adjoint $\widetilde{h}\colon s\mathbf{Set} \to N(\mathbf{Cat})$, defined by the following universal property: for all morphsims $f_\bullet\colon X_\bullet \to Y_\bullet$ of simplicial sets, where $Y_\bullet$ is the nerve of some (small) category, there exists a unique simplicial map $\widetilde{f}_\bullet\colon \widetilde{h}X_\bullet \to Y_\bullet$ where $\widetilde{h}_\bullet X_\bullet$ is the nerve of a category such that the diagram



in $s\mathbf{Set}$ commutes.[21] Thus $\widetilde{h}$ induces a natural bijection $s\mathbf{Set}(\widetilde{h}X_\bullet, Y_\bullet) \cong s\mathbf{Set}(X_\bullet, Y_\bullet)$.

By the isomorphism $Ho\colon N(\mathbf{Cat}) \xrightarrow{\cong} \mathbf{Cat}$, by construction, there is an adjunction

$$h\colon s\mathbf{Set} \rightleftarrows \mathbf{Cat} : N$$

where $h = Ho \circ \widetilde{h}$. More explicitly, the map $\widetilde{h}$ is required to satisfy relations when restricted to X_0, X_1 and X_2. In particular, this gives $hX_\bullet$ is then the small category with objects X_0, and morphisms freely generated by the set X_1 modulo commutativity relations in X_2.

Example 2.3.9 ($h\Delta^n$). Noting that $N[\![n]\!] = \Delta^n$, the universal property defining $\widetilde{h}$ is satisfied by taking $\widetilde{h}$ to be the identity map on Δ^n thus, $h\Delta^n = Ho\Delta^n \cong [\![n]\!]$.

[20] As the notation $\pi_{\leq 1}(W)$ suggests, the fundamental groupoid of a space W contains information about both the fundamental group $\pi_1(W)$ and the path components $\pi_0(W)$ of W. Namely, for some $w \in W$, $\pi_{\leq 1}(W)(w, w) = \pi_1(W, w)$ and the set of isomorphism classes of objects of $\pi_{\leq 1}(W)$ is $\pi_0(W)$.

[21] Note that the homotopy category $hX_\bullet$ for a simplicial set $X_\bullet$ is defined uniquely up to (a unique) isomorphism, and that the assignment is functorial. The argument for uniqueness is the same as the one showing uniqueness of initial objects up to (a unique) isomorphism. This definition is inspired by Theorem 2.3.6 and Corollary 2.3.7 in [15].

Example 2.3.10 (Inner Horn)**.** For Λ^n_k an inner horn and a map $\Lambda^n_k \to N(\mathbf{C})_\bullet$, to the nerve of a category $\mathbf{C}$, by the fact that there is a unique way of composing a string of n-composable arrows in $\mathbf{C}$, there exists a unique map $\Delta^n \to N(\mathbf{C})_\bullet$ extending it. In other words, the universal property defining $\widetilde{h}$ is satisfied by the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, thus $h\Lambda^n_k = Ho\Delta^n \cong [\![n]\!]$.

Proposition 2.3.11 (Theorem on pg. 19 in [12])**.** If $X_\bullet$ is a quasi category then $hX_\bullet \cong Ho(X_\bullet)$.

Sketch of Proof. (Variation of Proof in [12]) Consider the simplicial map $X_\bullet \to N(Ho(X_\bullet))$ that acts as the identity on 0-simplices and maps a 1-simplex $e \in X_1$ to the class $[e]$ and all higher simplices determined by the map on 1-simplices using Remark 1.1.7.[22] It suffices to show that this map also satisfies the universal property in Definition 2.3.8. So let a category $\mathbf{C}$ and a diagram

$$\begin{array}{ccc} X_\bullet & \xrightarrow{f_\bullet} & N(\mathbf{C}) \\ {\scriptstyle NHo_{X_\bullet}}\downarrow & & \\ NHoX_\bullet & & \end{array}$$

be given. But then, the composite

$$NHoX_\bullet \xrightarrow{NHo(f_\bullet)} NHoN(\mathbf{C}) \xrightarrow{\cong} N(\mathbf{C})$$

of the induced map and N applied to the canonical isomorphism $HoNX_\bullet \xrightarrow{\cong} X_\bullet$, gives a unique filler. □

2.4 Characterisation of Nerves

In fact, Example 2.3.10 can be taken even further, it can be used to completely characterise the image $N(\mathbf{Cat}) \subseteq s\mathbf{Set}$ up to isomorphism of simplicial sets.

Theorem 2.4.1 (Nerve Theorem; Proposition 1.1.2.2 in [16])**.** A simplicial set $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to the nerve of a category if and only if all inner horns in $X_\bullet$ have unique fillers.

Proof. ($\Longrightarrow$) If $X_\bullet \cong N(\mathbf{C})_\bullet$ for some small category $\mathbf{C}$, then the the existence of a unique filler for a given inner horn Λ^n_k and diagram of solid arrows

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & N(\mathbf{C})_\bullet \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

is precisely the universal property defining $h\Lambda^n_k$, as seen in Example 2.3.10.[23]

($\Longleftarrow$) Conversely suppose the relevant condition is satisfied by the simplicial set $X_\bullet$, define the category $\mathbf{C}$ with

- objects $\mathrm{obj}(\mathbf{C}) := X_0$.
- morphisms $\mathbf{C}(c, c') := \{e \in X_1 \mid d_1(e) = c \text{ and } d_0(e) = c'\}$.
- composition of a pair of morphisms $f \colon c \to c'$ and $g \colon c' \to c''$, defined to be the face $d_1(\sigma) := g \circ f$ of the unique filler σ of horn $\Lambda^2_1 \xrightarrow{(g,-,f)} X_\bullet$ in $X_\bullet$.
- the identity on an object $x \in X_0$ defined to be $s_0(x)$.

[22] The map $X_\bullet \to N(Ho(X_\bullet))$ will be the unit of the adjunction $h \dashv N$ applied to $X_\bullet$ once the proof is complete.

[23] It was essentially unnecessary to introduce the homotopy category for the purpose of this proof. However, it will serve as motivation for further chapters.

Then if $f\colon c\to c'$, $g\colon c'\to c''$ and $h\colon c''\to c'''$ are morphisms in $\mathbf{C}$, there are inner horns

$$\Lambda^2_1 \xrightarrow{(g,-,f)} X_\bullet \quad\text{and}\quad \Lambda^2_1 \xrightarrow{(h,-,g)} X_\bullet$$

in $X_\bullet$, which by hypothesis, admit unique fillers σ and ω, respectively. Thus there are inner horns

$$\Lambda^2_1 \xrightarrow{(h,-,d_1(\sigma)=g\circ f)} X_\bullet \quad\text{and}\quad \Lambda^2_1 \xrightarrow{(d_1(\omega)=h\circ g,-,f)} X_\bullet$$

in $X_\bullet$, which again by the unique inner horn filling condition, admit unique fillers σ' and ω', respectively, for which $d_1(\omega')=h\circ(g\circ f)$ and $d_1(\sigma')=(h\circ g)\circ f$. Then the triples $(\omega,\sigma',-,\sigma)$ and $(\omega,\omega',-,\sigma)$ are compatible by inspection, and there are inner horns

$$\Lambda^3_2 \xrightarrow{(\omega,\sigma',-,\sigma)} X_\bullet \quad\text{and}\quad \Lambda^3_2 \xrightarrow{(\omega,\omega',-,\sigma)} X_\bullet$$

in $X_\bullet$ with unique fillers Ω and Σ, respectively. Thus by the simplicial identity $d_1\circ d_1=d_1\circ d_2$ and the unique inner horn filling condition (middle equality),[24]

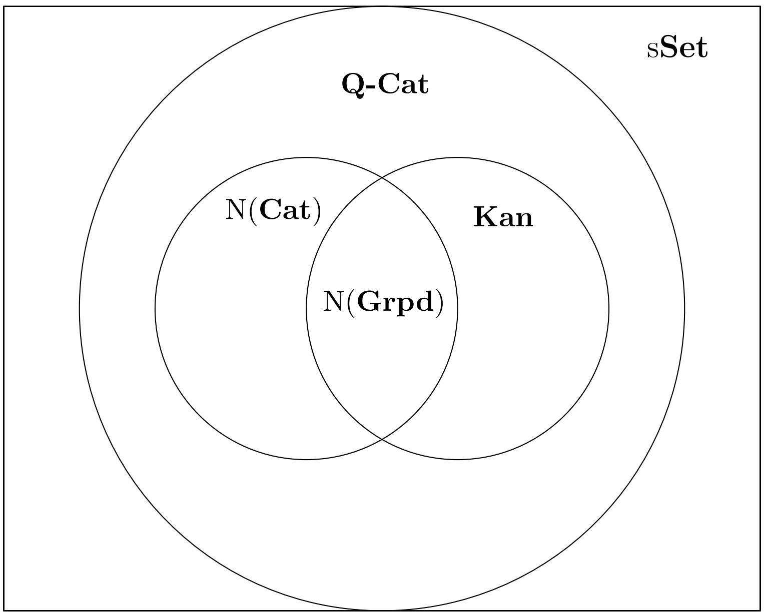
$$h\circ(g\circ f)=d_1(\omega')=d_1(d_1(\Omega))=d_1(d_2(\Omega))=d_1(d_2(\Sigma))=d_1d_1(\Sigma)=d_1(\sigma')=(h\circ g)\circ f$$

and so $\mathbf{C}$ indeed defines a category and a simplicial map $\delta_\bullet\colon X_\bullet\to N(\mathbf{C})_\bullet$ has been constructed. It is required to show that this map is an isomorphism of simplicial sets. It will be shown by induction on $n\geq 0$ that the maps $\delta_n\colon X_n\to N(\mathbf{C})_n$ are bijections for all $n\geq 0$. The base cases $n=0$ and $n=1$ are are clear from construction. Assume the claim is true for $n=m-1$ and let $0<i<m$ be given. Then δ_m is the composite of maps

$$X_m \xrightarrow{\cong} s\mathbf{Set}(\Delta^m,X_\bullet) \xrightarrow{\cong} s\mathbf{Set}(\Lambda^m_i,X_\bullet) \xrightarrow{\delta_\bullet\circ-} s\mathbf{Set}(\Lambda^m_i,N(\mathbf{C})_\bullet) \xrightarrow{\cong} s\mathbf{Set}(\Delta^m,X_\bullet) \xrightarrow{\cong} N(\mathbf{C})_m$$

where the first and last maps are isomorphisms by Lemma 2.1.4, the middle map is an isomorphism by inductive hypothesis, the second map is the map induced by the inclusion $\Lambda^m_i\hookrightarrow\Delta^m$ and is an isomorphisms by the unique inner horn filling condition, and the remaining map is the inverse of the map induced by the inclusion $\Lambda^m_i\hookrightarrow\Delta^m$ (which is an isomorphism for the same reason). $\square$

Up to isomorphism the equivalent statement in Theorem 2.4.1 is a reformulation of the definition of an ordinary small category. All in all, the full subcategories of s**Set** defined in terms of (weakening of) Kan conditions and the image of the nerve functor are summarised schematically by the Venn diagram



in which **Q-Cat** denotes the full subcategory of quasi categories and **Grpd**, the category of groupoids.

[24]Here it may be observed that the simplicial identities translate to "simplex chasing" arguments, such as that in the proof of Proposition 2.3.2, in low dimensions.

3 The Dold-Kan Correspondence

The main goal of this chapter is to demonstrate the equivalence between the category of non-negatively graded chain complexes over abelian groups and simplicial abelian groups. This fact has previously been briefly remarked on and this chapter outlines its proof. The construction in this Chapter may be generalised by replacing the category **Ab** of abelian groups with **Mod**(R) the category of (right or left) modules over a commutative ring R (with unit), or in fact, even more generally, with an arbitrary abelian category (see §5.5 pg. 307 in [27] for definition).

3.1 Normalised and Degenerate Complexes

Definition 3.1.1 (Normalised Complex; Definition 2.3 in [23] and Definition 8.3.6 in [29])**.** For $A_\bullet$, a simplicial abelian group, define the **normalised chain complex** $N_\star(A_\bullet)$ to be the chain complex,[25] defined by

$$N_0(A_\bullet) := A_0 \quad \text{and} \quad N_n(A_\bullet) := \bigcap_{i=0}^{n-1} \ker(d_i \colon A_n \to A_{n-1}) \subseteq A_n$$

for $n \geq 1$, with boundary maps

$$\partial_n^N := (-1)^n d_n \colon N_n(A_\bullet) \to N_{n-1}(A_\bullet).$$

Since by the simplicial identity $d_i \circ d_{i+1} = d_i \circ d_i$, the normalised complex $N_\star(A_\bullet)$ is indeed a chain complex. Furthermore, the assignment $A_\bullet \mapsto N_\star(A_\bullet)$ defines a functor from simplicial abelian groups to the category of non-negatively graded chain complexes $Ch_{\geq 0}(\mathbf{Ab})$ over abelian groups.

Example 3.1.2 ($N_\star(\mathbf{B}A_\bullet)$; Example 2.4 in [23])**.** For A an abelian group and $\mathbf{B}A$, the one object groupoid (as seen in Example 2.1.2) associated to A, the normalised chain complex $N_\star(\mathbf{B}A_\bullet)$ of its nerve $\mathbf{B}A_\bullet$, satisfies

- $N_0(\mathbf{B}A_\bullet) \cong \mathbf{B}A_0 = 0$.
- $N_1(\mathbf{B}A_\bullet) = \ker(d_0 \colon \mathbf{B}A_1 \to \mathbf{B}A_0 = 0)$, thus $N_1(\mathbf{B}A_\bullet) = \mathbf{B}A_1 \cong A$.
- for $n \geq 2$, that if $a = (a_1, \ldots, a_n) \in N_n(\mathbf{B}A_\bullet) \subseteq \mathbf{B}A_n \cong A^{\times n}$, then in particular, $a \in \ker(d_0)$, namely $d_0(a) = (a_2, \ldots, a_n) = (0, \ldots, 0)$ and since $a \in \ker(d_1)$, $a_1 + a_2 = 0$, thus $N_n(\mathbf{B}A_\bullet) = 0$.

Thus the n^{th} homology group of this complex is the abelian group A, if $n = 1$, and trivial, if otherwise. In fact this is also true of the homotopy groups for the Kan complex $\mathbf{B}A_\bullet$ (see Theorem 8.3.8 in [29] for details on this relation).

Definition 3.1.3 (Degenerate Complex; Definition 2.5 in [23] and Proposition 2.2 in [18])**.** For $A_\bullet$ a simplicial abelian group, define the **degenerate complex** $D_\star(A_\bullet)$ to be the subcomplex of the Moore complex $C_\star(A_\bullet)$, generated by degeneracies, that is with

$$D_n(A_\bullet) := \sum_{i=0}^{n-1} \operatorname{img}(s_i \colon A_{n-1} \to A_n)$$

where the empty sum is taken to be zero, and with the same differentials as $C_\star(A_\bullet)$. In particular, for $a \in A_{n-1}$,

$$\partial_n^D(s_j a) := \partial_n^C(s_j a) = \sum_{i=0}^{n} (-1)^i d_i s_j(a) = \sum_{i \neq j, j+1} (-1)^i s_j(d_i a)$$

where the last equality is by the simplicial identities, is a sum of $n-1$ degenerate simplices.

[25]The notation $N_\star$ is not to be confused with the notation N for the nerve, as it should be noted that the latter lacks a subscript.

Lemma 3.1.4 (Canonical Splitting; Lemma 2.9 in [18])**.** For $A_\bullet$ a simplicial abelian group, the inclusions $N_\star(A_\bullet) \hookrightarrow C_\star(A_\bullet) \hookleftarrow D_\star(A_\bullet)$ induce an isomorphism

$$N_\star(A_\bullet) \oplus D_\star(A_\bullet) \to C_\star(A_\bullet).$$

Proof. Omitted. See [9]. □

Thus the non-degenerate complex is the quotient $N_\star(A_\bullet) \cong C_\star(A_\bullet)/D_\star(A_\bullet)$.

3.2 Proof of the Correspondence

Definition 3.2.1 (Definition 3.1 in [23] and Lemmas 2.6 and 2.7 in [18])**.** For $Z_\star \in \mathrm{obj}(Ch_{\geq 0}(\mathbf{Ab}))$, define the simplicial abelian group with m-simplices the direct sum

$$\Gamma(Z_\star)_m := \bigoplus_{[\![m]\!] \twoheadrightarrow [\![n]\!]} Z_n,$$

indexed over all surjections with source $[\![m]\!]$ in the simplex category. For $f\colon [\![m]\!] \twoheadrightarrow [\![n]\!]$ an indexing surjection, and $g\colon [\![l]\!] \to [\![m]\!]$, the composite $f \circ g\colon [\![l]\!] \to [\![n]\!]$ may be factored uniquely using Lemma 1.1.2, as

$$f \circ g = d^{i_1} \circ \cdots \circ d^{i_h} \circ s^{j_1} \circ \cdots \circ s^{j_k}.$$

Define

$$(Z_n \to Z_{n-h}) := \begin{cases} 1, & h = 0 \\ (-1)^m \partial^C_m, & h = k = 1 \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

then the inclusion $Z_{n-h} \hookrightarrow \Gamma(Z_\star)_l$ into the factor indexed by $s^{j_1} \circ \cdots \circ s^{j_k}\colon [\![l]\!] \twoheadrightarrow [\![n-h]\!]$, gives by composition with the map labelled (3.1), a map $Z_n \to \Gamma(Z_\star)_l$. Finally, taking the direct sum of such maps over all surjections $f\colon [\![m]\!] \twoheadrightarrow [\![n]\!]$ gives a map

$$g^*\colon \Gamma(Z_\star)_m \to \Gamma(Z_\star)_l$$

in particular, $(d^i)^*$ and $(s^i)^*$ are defined as being the face and degeneracy maps for this simplicial abelian group, respectively. As f was a surjection, the map $1^*\colon \Gamma(Z_\star)_m \to \Gamma(Z_\star)_m$ induced by the identity on $[\![m]\!]$, is the identity map, and for another map $h\colon [\![p]\!] \to [\![l]\!]$ in $\mathbf{\Delta}$, $f \circ g \circ h$ may also be uniquely decomposed into cofaces and codegeneracies, giving a commutative diagram

$$\begin{array}{ccccc} [\![p]\!] & \xrightarrow{h} & [\![l]\!] & \xrightarrow{g} & [\![m]\!] \\ \big\downarrow & & \big\downarrow & & \big\downarrow f \\ [\![q]\!] & \longrightarrow & [\![n-h]\!] & \longrightarrow & [\![n]\!] \end{array}$$

in $\mathbf{\Delta}$ where the unlabelled maps are unique decompositions of $f \circ g \circ h$ and $f \circ g$ into cofaces and codegenarcies. By commutativity, repeating the construction shows $(g \circ h)^* = h^* \circ g^*$, and $\Gamma(Z_\star)_\bullet$ is indeed a contra-variant functor from the simplex category to abelian groups. For a map $\gamma_\star\colon Z_\star \to Z'_\star$ of chain complexes, there is a map

$$\Gamma(\gamma_\star)_\bullet\colon \Gamma(Z_\star)_\bullet \to \Gamma(Z'_\star)_\bullet$$

simply defined as $\gamma_\star$ on each factor. Thus Γ defines a functor $Ch_{\geq 0}(\mathbf{Ab}) \to s\mathbf{Ab}$.

Example 3.2.2 ($\Gamma(N_\star(\mathbf{B}A_\bullet))_{\bullet'}$). For the normalised chain complex $N_\star(\mathbf{B}A_\bullet)$ of the simplicial abelian group $\mathbf{B}A_\bullet$ associated to the abelian group A, there is an isomorphism

$$\Gamma(N_\star(\mathbf{B}A_\bullet))_m = \bigoplus_{[\![m]\!]\twoheadrightarrow[\![n]\!]} N_n(\mathbf{B}A_\bullet) \cong \bigoplus_{[\![m]\!]\twoheadrightarrow[\![1]\!]} A$$

by the computation in Example 3.1.2. Moreover, as there are exactly m surjections $[\![m]\!] \twoheadrightarrow [\![1]\!]$ (each map is determined by the minimal integer $0 < i \leq m$ such that $i \mapsto 1$), $\Gamma(N_\star(\mathbf{B}A_\bullet))_{\bullet'} \cong \mathbf{B}A_\bullet$.

Thus the simplicial abelian group $\mathbf{B}A_\bullet$ may be recovered from the complex $N_\star(\mathbf{B}A_\bullet)$ up to isomorphism, by applying Γ. This may be viewed as an application of one half of the correspondence.

Theorem 3.2.3 (Dold-Kan Correspondence; Theorem 2.7 in [23] and Theorem 2.5 in [18]). The functors $s\mathbf{Ab} \underset{\Gamma}{\overset{N_\star}{\rightleftarrows}} Ch_{\geq 0}(\mathbf{Ab})$ are quasi-inverses of each other giving an equivalence of categories $s\mathbf{Ab} \simeq Ch_{\geq 0}(\mathbf{Ab})$.

Proof. This will follow immediately from Lemmas 3.2.4 and 3.2.5. □

Lemma 3.2.4 ($\Gamma N_\star \cong 1$; Definition 3.2 and Lemma 3.3 in [23] and §2.5 in [18]). There is a natural isomorphism of functors

$$\Phi\colon \Gamma \circ N_\star \implies 1_{s\mathbf{Ab}}.$$

Proof. For $A_\bullet$ a given simplicial abelian group, define

$$\Phi_m(A_\bullet)\colon \Gamma(N_\star(A_\bullet))_m = \left(\bigoplus_{[\![m]\!]\twoheadrightarrow[\![n]\!]} N_n(A_\bullet)\right) \to A_m$$

to be the map that acts on the factor corresponding to the index $f\colon [\![m]\!] \twoheadrightarrow [\![n]\!]$, as the composition

$$N_n(A_\bullet) \hookrightarrow A_n \xrightarrow{A_\bullet(f)} A_m.$$

Note that this is indeed natural in $A_\bullet$. The claim that it is a natural isomorphism will be shown by induction on m. The base case being that the map $\Phi_0(A_\bullet)\colon \Gamma(N_\star(A_\bullet))_0 = N_0(A_\bullet) \to A_0$ is an isomorphism, holds as $A_\bullet(1_{[\![0]\!]})$ is the identity map on A_0. Assume the claim holds for $m-1$.

(Surjective) By Lemma 3.1.4, there is an isomorphism, $A_m \cong N_m(A_\bullet) \oplus D_m(A_\bullet)$. But notice

$$N_m(A_\bullet) \subseteq \operatorname{img}(\Phi_m(A_\bullet))$$

as the factor corresponding to the identity. As Φ_{m-1} is surjective, upon taking degeneracies,

$$D_m(A_\bullet) = \sum_{i=0}^{m-1} \operatorname{img}(s_i) \subseteq \operatorname{img}(\Phi_m(A_\bullet))$$

thus Φ_m is epic.

(Injective) First note that the map is injective on the factor corresponding to the identity on $[\![m]\!]$, for it is the inclusion $N_m(A_\bullet) \hookrightarrow A_m$. Suppose now $(a_f)_f \in \ker(\Phi_m(A_\bullet)) \subseteq \bigoplus_{f\colon [\![m]\!]\twoheadrightarrow[\![n]\!]} N_n(A_\bullet)$. To each indexing epimorphism $f\colon [\![m]\!] \twoheadrightarrow [\![n]\!]$, assign a map

$$\phi_f\colon [\![n]\!] \to [\![m]\!]$$
$$\phi_f(i) = max\{j \in [\![m]\!] \mid f(j) = i\}.$$

Using this define the partial ordering

$$f \leq f' \iff \phi_f(i) \leq \phi_{f'}(i) \quad \text{for each } i,$$

on surjections in $\mathbf{\Delta}([\![m]\!], [\![n]\!])$. Suppose towards the contrary that $(a_f)_f \neq 0$, then there exists a $g \neq 1_{[\![m]\!]}$ such that $a_g \neq 0$. Set g to be the surjection with this property which is maximal with respect to the partial ordering above. Then by naturality, the square

$$\begin{array}{ccc} \Gamma(N_\star(A_\bullet))_m & \xrightarrow{\phi_g^*} & \Gamma(N_\star(A_\bullet))_n \\ {\scriptstyle \Phi_m(A_\bullet)}\downarrow & & \downarrow{\scriptstyle \Phi_n(A_\bullet)} \\ A_m & \xrightarrow[A_\bullet(\phi_g)]{} & A_n \end{array}$$

commutes, and in particular, as $(a_f)_f \in \ker(\Phi_m(A_\bullet))$, one has $\Phi_n(A_\bullet)\left(\phi_g^*\,(a_f)_f\right) = 0$. But then by the assumed inductive hypothesis $\phi_g^*\,(a_f)_f = 0$. However, as by construction ϕ_g is a section of g (i.e. $g \circ \phi_g = 1_{[\![n]\!]}$), the factor of $\phi_g^*\,(a_f)_f$ indexed by the identity map is a_g. Thus $a_g = 0$, so there is a contradiction and Φ_m is injective. □

Lemma 3.2.5 ($N_\star\Gamma\cong 1$; Lemma 3.4 in [23] and §2.6 in [18])**.** There is a natural isomorphism of functors

$$\Psi\colon N_\star \circ \Gamma \implies 1_{Ch_{\geq 0}(\mathbf{Ab})}$$

Proof. Define $\widetilde{\Psi}$ by the level-wise inclusions

$$\widetilde{\Psi}_m(Z_\star)\colon N_m(\Gamma(Z_\star)_\bullet) \hookrightarrow C_m(\Gamma(Z_\star)_\bullet) = \Gamma(Z_\star)_m$$

for a given chain complex $Z_\star$, the claim is that the image of $\widetilde{\Psi}$ is the factor Z_m (labelled by the identity map). Restricting the codomain of $\widetilde{\Psi}$ to its image to obtain Ψ, will then imply the result. Assess the surjections in the indices of $\Gamma(Z_\star)_m = \bigoplus_{[\![m]\!]\twoheadrightarrow[\![n]\!]} Z_n$.

($m = n$) When $n = m$, the only index is the identity map indexing Z_m. Now as

$$N_m(\Gamma(Z_\star)_\bullet) = \bigcap_{i=0}^{m-1} ker\left((d^i)^*\colon \Gamma(Z_\star)_m \to \Gamma(Z_\star)_{m-1}\right),$$

and by equation 3.1, the map $Z_m \to Z_{m-1}$ constructed for computing $(d^i)^*$ (with $f = 1_{[\![m]\!]}$), is the constant map at zero (as $h = 1$ and $k = 0$), one has $N_m(\Gamma(Z_\star)_\bullet) \supseteq Z_m$ in other words, $img\left(\widetilde{\Psi}_m(Z_\star)\right) \supseteq Z_m$.

($m > n$) When $m > n$, a surjection $f\colon [\![m]\!] \twoheadrightarrow [\![n]\!]$, admits a factorisation $f = f' \circ s^i$ for some map $f'\colon [\![m-1]\!] \to [\![n]\!]$ in $\mathbf{\Delta}$. Thus the factor corresponding to the index f lies in $D_m(\Gamma(Z_\star)_\bullet)$. But by Lemma 3.1.4, $\Gamma(Z_\star)_m \cong N_m(\Gamma(Z_\star)_\bullet) \oplus D_m(\Gamma(Z_\star)_\bullet)$ and so the image of $\widetilde{\Psi}_m(Z_\star)$ is Z_m. □

Thus the Dold-Kan correspondence (Theorem 3.2.3) has been established.

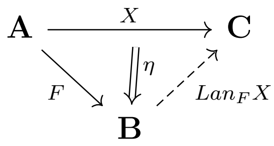
4 Geometric Realisation

Some of the theory of simplicial sets is preserved by the geometric realisation functor. Due to this, as the name suggests, the geometric realisation provides geometric intuition for simplicial sets. Recall the functor $|\Delta^\bullet|$ defined in Example 1.2.6. The geometric realisation of a simplicial set is a functor $|\cdot|\colon s\mathbf{Set} \to \mathbf{Top}$ whose restriction to $N(\mathbf{\Delta})$ factors this cosimplicial space as $[\![n]\!] \overset{N}{\mapsto} \Delta^n \overset{|\cdot|}{\mapsto} |\Delta^n|$. In the following sections it will be defined as an extension to a functor that carries the simplicial set Δ^n to the standard geometric n-simplex $|\Delta^n|$.

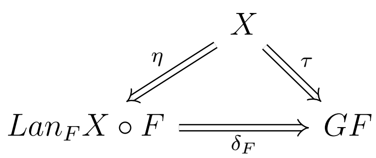
Notation 4.0.1. From now on in, for brevity, the bullet in the subscript of the notation $X_\bullet$ for simplicial sets, indicating the filtration of sets, has been omitted whenever it is unnecessary.

4.1 Yoneda Extensions

Definition 4.1.1 (Left Kan Extension; Ch. X, §3 in [17])**.** For categories $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, and functors



depicted by single lined solid arrows, the **left Kan extension** $Lan_F X$, if it exists, is a functor $\mathbf{B} \to \mathbf{C}$ depicted by the dashed arrow together with a natural transformation $\eta\colon X \implies Lan_F X \circ F$ which is universal in the sense that for any pair consisting of a functor $G\colon \mathbf{B} \to \mathbf{C}$ and a natural transformation $\tau\colon X \implies GF$,[26] there is a unique natural transformation $\delta\colon Lan_F X \implies G$ such that the diagram

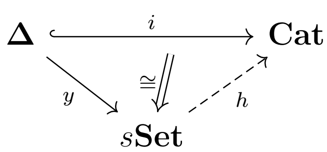


in the functor category $[\mathbf{A}, \mathbf{C}]$ commutes.

Example 4.1.2 (Homotopy Category; Example 1.10 in [10])**.** For $i\colon \mathbf{\Delta} \hookrightarrow \mathbf{Cat}$ the inclusion functor and

$$y := N|_{\mathbf{\Delta}}\colon \mathbf{\Delta} \to s\mathbf{Set}$$

the **Yoneda embedding**, the homotopy category functor $h\colon s\mathbf{Set} \to \mathbf{Cat}$ defined in Definition 2.3.8, is the left Kan extension $Lan_y i$



of the inclusion i along the Yoneda embedding y, together with the natural isomorphism η provided by inverting the counit of the adjunction $h \dashv N$ and then restricting it to $\mathbf{\Delta}$. For any other pair consisting of a functor $G\colon s\mathbf{Set} \to \mathbf{Cat}$ and a natural transformation $\tau\colon i \implies G \circ y$, the functor $N \circ G\colon s\mathbf{Set} \to N(\mathbf{Cat})$ gives by the universal property that was used to define $\widetilde{h}$ (in Definition 2.3.8), a unique natural transformation $\delta\colon h \implies G$ making the relevant diagram commute by construction as $y = N|_{\mathbf{\Delta}}$.

[26]The left Kan extension will be used in reffering to both the functor as well as the pair consisting of the functor and the natural transformation.

Remark 4.1.3 (Definition 1.8 in [10])**.** Left Kan extensions along Yoneda embeddings are called **Yoneda extensions** and are guaranteed to exist as a consequence of Theorem 2.1.5, when the category $\mathbf{C}$ (with notation borrowed from Definition 4.1.1) is cocomplete (i.e. has colimits over all diagrams of shape $\mathbf{J}$, where $\mathrm{obj}(\mathbf{J})$ is a finite set), as was the case for **Cat** in the above example. There is a formulaic way of computing the Yoneda extension in such a situation.

The Yoneda extension $Lan_y|\Delta^\bullet|$ of the functor $|\Delta^\bullet|$ from Example 1.2.6 is defined to be the geometric realisation of a simplicial set. The formulaic construction mentioned in Remark 4.1.3 is now carried out to give an explicit description of the geometric realisation functor.

4.2 Realisation as a Coend

All definitions in this section are specified towards the construction of the geometric realisation functor $|\cdot|\colon s\mathbf{Set} \to \mathbf{Top}$. In addition to providing for motivation, defining the geometric realisation as a certain colimit, will serve useful in showing that it is a left adjoint.

Note (Bibliographical Note). This section has been taken almost verbatim from Chapter 4 in [24]. Thus only material taken from elsewhere has been referenced.

Definition 4.2.1 (Copower)**.** If X is a simplicial set, then the **copower** of X_m *by* the topological space $|\Delta^n|$, is the space defined to be the disjoint union

$$X_m \cdot |\Delta^n| := \coprod_{x \in X_m} |\Delta^n|$$

of copies of $|\Delta^n|$ labelled by elements of X_m. Equivalently, it is the Cartesian product $X_m \times |\Delta^n|$, where X_m is given the discrete topology.

If $f\colon [\![n]\!] \to [\![m]\!]$ is an n-simplex in Δ^m_n, then f induces a map $X(f)\colon X_m \to X_n$ and a map $|\Delta^\bullet|(f) := B(f)\colon |\Delta^n| \to |\Delta^m|$ sending a point $(t_0, \dots, t_n) \in |\Delta^n| \subseteq \mathbb{R}^{n+1}$ to the point $\left(t'_j\right)_j$ in $|\Delta^m|$ where

$$t'_j := \sum_{f(i)=j} t_i$$

where in, the empty sum taken to be zero.[27] Thus f induces continuous maps

$$f_* := 1 \times B(f)\colon X_m \cdot |\Delta^n| \to X_m \cdot |\Delta^m| \quad \text{and} \quad f^* := X(f) \times 1\colon X_m \cdot |\Delta^n| \to X_n \cdot |\Delta^n|.$$

Definition 4.2.2 (Geometric Realisation)**.** For the diagram consisting of copowers $X_m \cdot |\Delta^n|$ as objects for all $m, n \in \mathbb{N}$ and continuous maps f^* and f_* for each $f \in \mathrm{mor}(\boldsymbol{\Delta})$, a **wedge** over this diagram is a topological space W, together with continuous maps $\delta_k\colon X_k \cdot |\Delta^k| \to W$ making the squares

$$\begin{array}{ccc} X_m \cdot |\Delta^n| & \xrightarrow{f_*} & X_m \cdot |\Delta^m| \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \delta_m} \\ X_n \cdot |\Delta^n| & \xrightarrow[\delta_n]{} & W \end{array}$$

commute. The **geometric realisation** $|X|$ of the simplicial set X is the wedge $\left(W, (\delta_k)_{k\in\mathbb{N}}\right)$ over this diagram which is universal in the sense that for any other wedge $\left(V, (\lambda_k)_{k\in\mathbb{N}}\right)$ over this diagram, there is a unique map $W \to V$ making the resulting diagram commute.[28] Additionally, from this universal property, a simplicial map $g\colon X \to Y$ induces a unique map $|g|\colon |X| \to |Y|$ making the geometric realisation into a functor $|\cdot|\colon s\mathbf{Set} \to \mathbf{Top}$.

[27] This formula was taken from Definition 1.1 in [18]

[28] More generally, this universal wedge is called the **tensor product** or **coend** of the functor $|\Delta^\bullet|$ and the simplicial space DX obtained by endowing each X_n with the discrete topology.

Remark 4.2.3. As each square in the above definition may be labelled by the map inducing it, the geometric realisation is equivalently the coequaliser (colimit) of the diagram

$$\coprod_{f\colon [\![n]\!]\to[\![m]\!]} X_m \cdot |\Delta^n| \underset{f^*}{\overset{f_*}{\rightrightarrows}} \coprod_{[\![n]\!]} X_n \cdot |\Delta^n|$$

As a map in $\mathbf{\Delta}$ can be decomposed into a composition of coface and codegenracy maps using Lemma 1.1.2, there is an explicit description of the realisation of a simplicial set obtained from analysing the commutativity relations from the coequaliser.

Definition 4.2.4 (Geometric Realization; Definition 4.3 in [4] and Introduction to Ch. III, §14 in [19])**.** The **geometric realisation** (or **realisation** for short) of a simplicial set X is defined to be the quotient space

$$|X| := \left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n|\right)/\sim$$

where for each n, X_n is equipped with the discrete topology and $|\Delta^n|$ is equipped with the subspace topology from $\mathbb{R}^{n+1}$. The equivalence relation defined by the rule that $(x,\sigma)\sim(y,\tau)$ iff

- $(x,\sigma)=(y,\tau)$ or
- $X(d^i)(x)=y$ and $\sigma=B(d^i)(\tau)$ or
- $x=X(s^j)(y)$ and $B(s^j)(\sigma)=\tau$

for some i, j. The class of the point $(x,\sigma)\in X_n\times|\Delta^n|$ under this equivalence relation is denoted by the symbol $|(x,\sigma)|$. A simplicial map $g\colon X\to Y$ of simplicial sets induces the continuous map

$$\begin{aligned}|g|\colon |X| &\to |Y|\\ |(x,\sigma)| &\mapsto |(g(x),\sigma)|\end{aligned}$$

between spaces $|X|$ and $|Y|$.

4.3 Examples

Example 4.3.1 ($|\Delta^0|$; Example 4.2 in [7])**.** As the simplicial Δ^0 consists of a unique n-simplex (constant map $[\![n]\!]\to[\![0]\!]$) for each n, the equivalence relation holds vacuously for each index of cofaces and codegeneracy maps $\Delta^0(d^i)$ and $\Delta^0(s^j)$ on the first component. Thus as the point $\{1\}=|\Delta^0|$ is the image under a sequence of codegeneracies $B(s^i)$'s of *any* point $(t_0,\dots,t_n)\in|\Delta^n|$ (by the constraint $t_0+\cdots+t_n=1$), the geometric realisation of Δ^0 is the one point space.

Example 4.3.2 ($|\Delta^k|$; pg. 8 in [24])**.** More generally, the geometric realisation of the simplicial set Δ^k is homeomorphic to the subspace of $\mathbb{R}^{n+1}$ consisting of the set of non-negative $(n+1)$-tuples whose coordinates sum to one, that has been denoted as $|\Delta^k|$. Until the homeomorphism has been established, let $|\Delta^k|$ exclusively denote this described subspace of $\mathbb{R}^{n+1}$. It suffices to show that $|\Delta^k|$ is also a universal wedge over the diagram in Definition 4.2.2. For this construct maps

$$\begin{aligned}\delta_l\colon \Delta^k_l\cdot|\Delta^l| &\to |\Delta^k|\\ (x,\sigma) &\mapsto |x|(\sigma)\end{aligned}$$

for each l. Then with the same notations as in Definition 4.2.2, for each $n,m\in\mathbb{N}$ and $f\in\Delta^m_n$, the squares

$$\begin{array}{ccc} \Delta^k_m \cdot |\Delta^n| & \xrightarrow{f_*} & \Delta^k_m \cdot |\Delta^m| \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle \delta_m} \\ \Delta^k_n \cdot |\Delta^n| & \xrightarrow[\delta_n]{} & |\Delta^k| \end{array}$$

commute. For another given wedge $\left(W, (\lambda_l)_{l\in\mathbb{N}}\right)$ over this diagram, the map $\lambda_k \colon \Delta^k_k \times |\Delta^k| \to W$ gives a map $\lambda_k(1_{[\![k]\!]}, -) \colon |\Delta^k| \to W$. With this, the triangles

$$\begin{array}{ccc} \Delta^k_l \times |\Delta^l| & \xrightarrow{\delta_l} & |\Delta^k| \\ & {\scriptstyle \lambda_l}\searrow & \big\downarrow{\scriptstyle \lambda_k(1_{[\![k]\!]}, -)} \\ & & W \end{array}$$

commute by the fact that W is a wedge.[29] Moreover $\lambda_k(1_{[\![k]\!]}, -)$ is the unique such map making the triangle with $l = k$ commute. Hence implying the result and justifying the conflicting notations.

Example 4.3.3 (Sphere; Definition 5.2 in [24]). Let $\partial\Delta^n$ be the subsimplicial set of Δ^n, generated by the $(n-1)$-simplices

$$\left\{ d^i \colon [\![n-1]\!] \to [\![n]\!] \;\middle|\; i \in \{0, \ldots, n\} \right\}.$$

That is, the simplicial set with m-simplices $(\partial\Delta^n)_m$ consisting of maps $f \colon [\![m]\!] \to [\![n]\!]$ that can be factored as $f = d^i \circ f'$ for some $f' \in \Delta^{n-1}_m$, thus the set of maps $f \colon [\![m]\!] \to [\![n]\!]$ that are *not* surjective. The simplicial set $\partial\Delta^n$ is called the **simplicial sphere** of dimension $n-1$. Let

- $\partial^i\Delta^n$ be the subsimplicial set of Δ^n generated by the $(n-1)$-simplex $d^i \colon [\![n-1]\!] \to [\![n]\!]$ of Δ^n.
- $\partial^i\partial^j\Delta^n$ be the subsimplicial set of Δ^n generated by the $(n-2)$-simplex $d^id^j \colon [\![n-2]\!] \to [\![n]\!]$ of Δ^n.

Then there as isomorphisms $\partial^i\Delta^n \cong \Delta^{n-1}$ and $\partial^i\partial^j\Delta^n \cong \Delta^{n-2}$. Due to the cosimplicial identities, $\partial^i\partial^j\Delta^n = \partial^{j+1}\partial^i\Delta^n$ for all $i \leq j$,[30] and the simplicial sphere $\partial\Delta^n$ may be expressed as the colimit over the diagram with pairs of inclusions

$$\begin{array}{ccc} & \partial^i\partial^j\Delta^n & \\ \swarrow & & \searrow \\ \partial^i\Delta^n & & \partial^{j+1}\Delta^n \end{array}$$

for all pairs i, j. Thus as the geometric realisation is also constructed as a certain colimit, the realisation of the colimit $\partial\Delta^n$ of this diagram, is the colimit of the realisation applied to this diagram, which by Example 4.3.2 gives that $|\partial\Delta^n|$ is homeomorphic to boundary of the space $|\Delta^n|$.

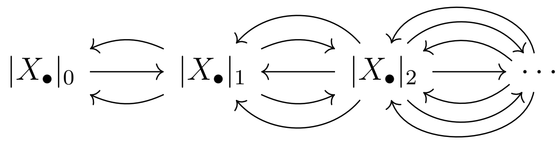
Example 4.3.4 (Horn; Definition 5.3 in [24]). Similarly, Λ^n_k may be constructed as a colimit over the same diagram as of that in Example 4.3.3, but excluding the components involving k. Thus giving by the same argument that $|\Lambda^n_k| \cong \partial|\Delta^n| \backslash \{(t_0, \ldots, t_n) \in |\Delta^n| : t_k = 0\}$ and implying the claim of Example 2.2.7.

Example 4.3.5 (Classifying Category; Ch. I, §1, Example 1.4 in [9]). The geometric realisation $|N(\mathbf{C})|$ of the nerve of a small category $\mathbf{C}$ is called the **classifying category** of $\mathbf{C}$ and is sometimes denoted $B(\mathbf{C})$.[31] For instance, when $\mathbf{C}$ is the category $[\![n]\!]$, Example 4.3.2 shows that $|N[\![n]\!]| \cong |\Delta^n|$ and when $\mathbf{C}$ is the groupoid $\mathbf{B}G$ associated to a discrete group G (as in Example 2.1.2), then the classifying category $B(\mathbf{B}G)$ is (a model for)[32] an Eilenberg-MacLane space $K(G, 1)$.

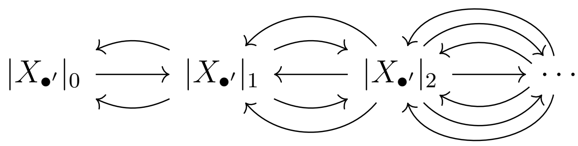
[29] This is a similar argument as of that in the proof of Yoneda's lemma.
[30] This holds without loss of generality.
[31] Sometimes $B(\mathbf{C})$ is used to denote the homotopy type of $|N(\mathbf{C})|$ instead.
[32] Here "a model for" refers the specific choice in the homotopy type of the classifying space.

For G a discrete group there is a homotopy equivalence $\Omega BG \simeq G$. For this reason the functor $B = |\cdot| \circ N$ is called a **delooping** of the discrete group G. See Proposition 4.66 and the example following it in [11], for an elegant proof of this.

Example 4.3.6 (Simplicial Space; Lemma, pg. 86 in [22])**.** For $X_{\bullet,\bullet'}$, a bisimplicial set, the level-wise realisation



of the simplicial sets $(X_\bullet)_n$ i.e. the composite $\mathbf{\Delta}^{\text{op}} \xrightarrow{(X_\bullet)_{\bullet'}} s\mathbf{Set} \xrightarrow{|\cdot|} \mathbf{Top}$, gives a simplicial space. Quillen showed in [22] that the realisation of this simplicial space[33] is homeomorphic to the space obtained by realising the simplicial set $(X_{n,n})_{n\in\mathbb{N}}$ and to the space obtained by realising the simplicial space



obtained by the level-wise realisation $\mathbf{\Delta}^{\text{op}} \xrightarrow{(X_{\bullet'})_\bullet} s\mathbf{Set} \xrightarrow{|\cdot|} \mathbf{Top}$ of other component instead.[34]

4.4 Results

Lemma 4.4.1 ($|\cdot| \dashv Sing$; Equation 4.4 in [24])**.** There is an adjunction

$$|\cdot|\colon s\mathbf{Set} \rightleftarrows \mathbf{Top} : Sing$$

between simplicial sets and topological spaces.

Proof. Let W be a given topological space. As any simplicial set is a colimit of the representable functors Δ^n (Theorem 2.1.5), and as the geometric realisation $|\cdot|$ is constructed as a colimit, it follows that the functor $|\cdot|$ commutes with colimits. Thus the functor $|\cdot|$ is completely determined by its value on the simplicial sets Δ^n and it is enough to show that

$$s\mathbf{Set}(\Delta^n, Sing(W)) \cong \mathbf{Top}(|\Delta^n|, W).$$

By Lemma 2.1.4, $s\mathbf{Set}(\Delta^n, Sing(W)) \cong Sing(W)_n$ and by definition, $Sing(W)_n = \mathbf{Top}(|\Delta^n|, W)$. □

Definition 4.4.2 (Non-degenerate; pg. 56, defined after Theorem 14.1 in [19])**.** An element $(x,\sigma) \in X_n \times |\Delta^n|$ is called **non-degenerate** if x is a non-degenerate n-simplex of X and $\sigma \in \text{int}(|\Delta^n|) := \{(t_0,\dots,t_n) \in |\Delta^n| \mid t_i > 0\ \forall i\}$.

Lemma 4.4.3 (Lemma 14.2 in [19])**.** Every element $(x,\sigma) \in X_n \times |\Delta^n|$ is equivalent to a unique non-degenerate element $(y,\tau) \in X_m \times |\Delta^m|$.

[33] Only the realisation of simplicial sets have been defined here, the realisation of simplicial spaces may be defined using a similar construction.

[34] This example was taken from https://mathoverflow.net/questions/97732/realization-of-a-bisimplicial-set.

Proof. Noting that every point in $|\Delta^n|$ may be written as an image under a sequence of compositions of coface maps $B(d^i)$ of a point in the interior of $|\Delta^n|$, and that every degenerate simplex in X_n may be written as a sequence under degeneracies $X(s^j)$ of a non-degerate simplex of X, define the maps

$$\begin{aligned} l, r\colon \left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n|\right) &\to \left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n|\right) \\ l\colon (z, B(d^{i_1} \circ \ldots \circ d^{i_l})(\alpha)) &\mapsto (X(d^{i_l} \circ \ldots \circ d^{i_1})(z), \alpha) \quad \text{where} \quad \alpha \in \mathrm{int}(|\Delta^n|) \\ r\colon (X(s^{j_k} \circ \ldots \circ s^{j_1})(z), \alpha) &\mapsto (z, B(s^{j_1} \circ \ldots \circ s^{j_k})(\alpha)) \quad \text{where } z \text{ is non-degenerate.} \end{aligned}$$

Now let $(x, \sigma) \in X_n \times |\Delta^n|$ be given, then by Lemma 1.1.2, $(y, \tau) := r \circ l(x, \sigma)$ is a unique non-degenerate $(m := n - k + l)$-simplex of X which is non-degenerate and equivalent to (x, σ). □

Theorem 4.4.4 ($|s\mathbf{Set}| \subseteq \mathbf{CW}$; Theorem 14.1 in [19])**.** The geometric realisation $|X|$ of a simplicial set is a CW-complex.

Proof. For X a simplicial set, let $X^{(n)}$ denote the n^{th} **skeleton** of X i.e. the subsimplicial set of X generated by simplices of dimension $\leq n$ and let $\underline{X_n}$ denote the set of non-degenerate simplices in X_n. Then by Lemma 4.4.3, for each $n \geq 1$, there is a pushout digram

$$\begin{array}{ccc} \coprod_{x \in \underline{X_n}} \partial\Delta^n & \longrightarrow & X^{(n-1)} \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{x \in \underline{X_n}} \Delta^n & \longrightarrow & X^{(n)} \end{array}$$

in $s\mathbf{Set}$, where the attaching map on the factor $x \in \underline{X_n}$ is the map

$$\partial\Delta^n \xrightarrow{(d_0(x), \ldots, d_n(x))} X^{(n-1)}.$$

As the realisation $|\cdot|$ is a left adjoint (or just by the fact that it was constructed as a colimit), it preserves pushouts and coproducts, thus applying the realisation $|\cdot|$ gives a pushout

$$\begin{array}{ccc} \coprod_{x \in \underline{X_n}} \partial|\Delta^n| & \longrightarrow & |X^{(n-1)}| \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{x \in \underline{X_n}} |\Delta^n| & \longrightarrow & |X^{(n)}| \end{array}$$

in $\mathbf{Top}$ (see Examples 4.3.2 and 4.3.3). Thus by homeomorphisms $\partial|\Delta^n| \cong S^{n-1}$ and $|\Delta^n| \cong D^n$, the space

$$|X| = colim\left(|X^{(0)}| \hookrightarrow |X^{(1)}| \hookrightarrow |X^{(2)}| \hookrightarrow \cdots\right)$$

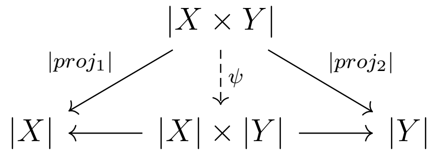
is a CW-complex. □

Remark 4.4.5. Note that this implies that a space W is not necessarily homeomorphic to the realisation $|Sing(W)|$ of its singular simplicial set $Sing(W)$ as the latter is necessarily a CW-complex whilst the former need not be.

The following proposition was proven by Milnor around 1957 (see [20]).

Theorem 4.4.6 (Theorem 14.3 in [19])**.** If X and Y are simplicial sets such that $|X| \times |Y|$ is a CW-complex, then there is a homeomorphism $|X \times Y| \cong |X| \times |Y|$.

Proof. Let $proj_i$ be the projection from $X \times Y$ to the i^{th} factor, for $i = 1, 2$, then as $|X| \times |Y|$ is the categorical product in the category **Top**, there is a unique dashed arrow making the diagram



where the unlabelled arrows are the canonical projections, commute. A continuous inverse to this unique map will now be constructed. Let $(x, \sigma) \in X_k \times |\Delta^k|$ and $(y, \tau) \in Y_l \times |\Delta^l|$ be given representatives for classes in $|X|$ and $|Y|$, respectively. By Lemma 4.4.3, there exist unique non-degenerate representatives $(x', \sigma') \in X_{k'} \times |\Delta^{k'}|$ and $(y', \tau') \in Y_{l'} \times |\Delta^{l'}|$. Say $\sigma' = (t_0, \ldots, t_{k'})$ and $\tau' = (\theta_0, \ldots, \theta_{l'})$. Set

$$\left(p_m := \sum_{i=0}^{m} t_i\right)_{m=0}^{k'} \quad \text{and} \quad \left(q_n := \sum_{i=0}^{n} \theta_i\right)_{n=0}^{l'}$$

to be the tuples of partial sums of the barycentric coordinates. Combine both tuples to form a $(j+1)$-tuple $0 < a_0 < \ldots < a_j = 1$ of (distinct) terms in $\{p_m\}_{m=0}^{k'} \cup \{q_n\}_{n=0}^{l'}$ rewritten in ascending order. From this, form the $(j+1)$-tuple

$$(\widetilde{a_i})_{i=0}^{j} \quad \text{where} \quad \widetilde{a_i} := a_i - a_{i-1} \quad \text{with} \quad a_{-1} := 0.$$

Thus $\sum_{i=0}^{j} \widetilde{a_i} = a_j = 1$ and $0 < \widetilde{a_i}$ for each i, hence $a := (\widetilde{a_0}, \ldots, \widetilde{a_j})$ is in the interior of $|\Delta^j|$. Now define tuples of all integers u such that $a_u \notin \{p_m\}_{m=0}^{k'}$ rewritten in ascending order $u_1 < \ldots < u_{j-k'}$ and similarly, all integers v such that $a_v \notin \{q_n\}_{n=0}^{j-l'}$ rewritten in ascending order $v_1 < \ldots < v_{j-l'}$. Then $\{u_i\}_{i=1}^{j-k'} \cap \{v_i\}_{i=1}^{j-l'} = \emptyset$ and $\sigma' = B(s^{u_1} \ldots s^{u_{j-k'}})(a)$ and $\tau' = B(s^{v_1} \ldots s^{v_{j-l'}})(a)$. Finally, with the same notations thus far, set

$$\begin{aligned} \varphi \colon |X| \times |Y| &\to |X \times Y| \\ (|(x, \sigma)|, |(y, \tau)|) &\mapsto |(X(s^{u_{j-k'}} \ldots s^{u_1})(x'), Y(s^{v_{j-l'}} \ldots s^{v_1})(y'), a)|. \end{aligned}$$

Then letting r be the map defined in Lemma 4.4.3,

$$\begin{aligned} |proj_1|(\varphi(|(x, \sigma)|, |(y, \tau)|)) &= |proj_1|\,|(X(s^{u_{j-k'}} \ldots s^{u_1})(x'), Y(s^{v_{j-l'}} \ldots s^{v_1})(y'), a)| \\ &= |(X(s^{u_{j-k'}} \ldots s^{u_1})(x'), a)| \\ &= |r\,(X(s^{u_{j-k'}} \ldots s^{u_1})(x'), a)| \\ &= |(x', B(s^{u_1} \ldots s^{u_{j-k'}})(a))| = |(x', \sigma')| = |(x, \sigma)| \end{aligned}$$

and similarly, $|proj_2|(\varphi(|(x, \sigma)|, |(y, \tau)|)) = |(y, \tau)|$ so that $\psi \circ \varphi = 1$. Now let $|(z_1, z_2, \omega)| \in |X \times Y|$, and let (z_1, z_2, ω) be a non-degenerate representative of $|(z_1, z_2, \omega)|$ then it follows that $r(z_i, \omega)$ is a non-degenerate representative for $|(z_i, \omega)|$, for $i = 1, 2$. Thus

$$\varphi(\psi(|(z_1, z_2, \omega)|)) = \varphi(|(z_1, \omega)|, |(z_2, \omega)|) = \varphi(|r(z_1, \omega)|, |r(z_2, \omega)|) = |(z_1, z_2, \omega)|$$

and so $\varphi \circ \psi = 1$. Finally, ψ is continuous by construction and φ is continuous for it is continuous on each cell of the CW-complex $|X| \times |Y|$. □

In particular, the realisation $|\cdot|$ as a functor to compactly generated Hausdorff spaces, preserves (finite) products.

Corollary 4.4.7 (Corollary 14.6 in [19])**.** If there is a multiplication $\mu\colon X \times X \to X$ on a simplicial set X, then the continuous map $|\mu| \circ \varphi\colon |X| \times |X| \to |X|$ (where φ is as above, but with X in place of Y) is a multiplication on $|X|$ in **Top**. For instance, the realisation of a simplicial monoid, group or abelian group is a topological monoid, group or abelian group, respectively.

Additionally, due to this proposition, homotopy between simplicial maps will give rise to homotopies between their respective realisations. Homotopy theory in the category of simplicial sets is now developed.

5 Simplicial Homotopy Theory

In this chapter, some of the homotopy theory of simplicial sets is developed with analogy to that of topological spaces. This is done purely combinatorially for Kan complexes, with the main source of reference being Peter May's book, [19]. Homotopies between 1-simplices and homology of simplicial sets were covered in Chapters 1 and 2. Similarly, homotopy between simplices, homotopy between simplicial maps, simplicial homotopy groups, fibre sequences and so on, may all be defined for Kan complexes in this way. However, first some terminology is fixed.

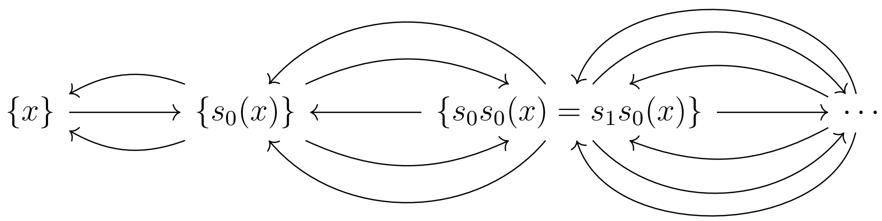
5.1 New Simplicial Sets from Old

Definition 5.1.1 (Pair and Triple; Definition 8.10 in [7] and Notation 3.5 in [19])**.** A **simplicial pair** (X, A) consists of a simplicial set X and a subsimplicial set A of X (as defined in Definition 2.2.1). A **simplicial triple** (X, A, A'), is a triple of simplicial sets such that there is a pair of successive inclusions $A' \subseteq A \subseteq X$ of subsimplicial sets.

- A simplicial pair (X, A) is called a **Kan pair** if both X and A are Kan complexes.
- A simplicial triple (X, A, A') is called a **Kan triple** if each of X, A and A' are Kan complexes.
- A **simplicial map of pairs** $(X, A) \to (Y, B)$ is a simplicial map $X \to Y$ which restricts to another simplicial map $A \to B$.
- A **simplicial map of triples** $(X, A, A') \to (Y, B, B')$ is a simplicial map $X \to Y$ which restricts to a simplicial map $A \to B$ which further restricts to a simplicial map $A' \to B'$.

The following example of a simplicial/Kan pair/triple will be most relevant.

Example 5.1.2 (Basepoint; Example 8.11 in [7])**.** Let X be a simplicial set and let $\langle x \rangle$ denote a **basepoint** of X, i.e. the subsimplicial set



of X consisting of an element x of X_0 along with all of its degeneracies. It follows from the simplicial identities that $\langle x \rangle_n$ is a singleton for each n and is thus canonically isomorphic to Δ^0. Moreover, the unique map $\Delta^n \to \langle x \rangle$ is the filler of any horn $\Lambda^n_k \to \langle x \rangle$ in $\langle x \rangle$, and so the simplicial pair $(X, \langle x \rangle)$ is a Kan pair if and only if X is a Kan complex. The simplicial pair (respectively Kan pair) $(X, \langle x \rangle)$ is called a **pointed simplicial set** (respectively **pointed Kan complex**). For a subsimplicial set A of X, containing the base-point $\langle x \rangle$, the simplicial triple (respectively Kan triple, if A is a Kan complex) $(X, A, \langle x \rangle)$ is called a **pointed simplicial pair** (respectively **pointed Kan pair**). The symbol $\langle x \rangle_n$ will be used to refer to both the singleton as well as the simplex it contains.

It is easy to see that (pointed) simplicial sets and (pointed) simplicial pairs form categories with full subcategories of (pointed) Kan complexes and (pointed) Kan pairs, respectively.

Remark 5.1.3. (Example 8.12 in [7]) If X is a Kan complex and (X, A) is a simplicial pair then it is not always the case that A is a Kan complex. Consider the map $F\colon \Delta^1 \to Sing|\Delta^1|$ that carries the functor $f\colon [\![n]\!] \to [\![1]\!]$ in Δ^1_n to the map $B(f)\colon |\Delta^n| \to |\Delta^1|$. Then F is (level-wise) injective as the functor $|\Delta^\bullet|$ is faithful. Now consider the simplicial pair $(Sing(|\Delta^1|), F(\Delta^1))$ where in, the simplicial set $F(\Delta^1)$, being the image of an injection, is isomorphic to Δ^1 and hence by Example 2.2.6, is not a Kan complex, however, by Example 2.2.7, the simplicial set $Sing(|\Delta^1|)$ is a Kan complex.

Example 5.1.4. $((\Delta^n, \partial\Delta^n))$ The natural inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ makes $(\Delta^n, \partial\Delta^n)$ a simplicial pair that is not always a Kan pair as Δ^1 is not a Kan complex (by Example 2.2.6).

Definition 5.1.5 (Functional Simplicial Set; Definition 6.4 in [19])**.** The **functional simplicial set** Y^X is the simplicial set with n-simplices $\left(Y^X\right)_n$ the set of maps $s\mathbf{Set}(X \times \Delta^n, Y)$. Recalling notation for the nerve from Definition 2.1.1, Y^X has face maps

$$\begin{aligned} d_i \colon s\mathbf{Set}(X \times \Delta^n, Y) &\to s\mathbf{Set}(X \times \Delta^{n-1}, Y) \\ f &\mapsto f \circ (1 \times N(d^i)) \end{aligned}$$

and degeneracies

$$\begin{aligned} s_i \colon s\mathbf{Set}(X \times \Delta^n, Y) &\to s\mathbf{Set}(X \times \Delta^{n+1}, Y) \\ f &\mapsto f \circ (1 \times N(s^i)) \end{aligned}$$

and in particular, $\left(Y^X\right)_0 \cong s\mathbf{Set}(X, Y)$.

Example 5.1.6 (X^{Δ^0})**.** Using Lemma 2.1.4, there are bijections $\left(X^{\Delta^0}\right)_n \cong s\mathbf{Set}(\Delta^n, X) \cong X_n$. Thus there is an isomorphism of simplicial sets $X^{\Delta^0} \cong X$.

For all simplicial sets Y, there is an adjunction $- \times Y \dashv (-)^Y$,[35] the bijection

$$s\mathbf{Set}(X \times Y, Z) \cong s\mathbf{Set}\left(X, Z^Y\right) \tag{5.1}$$

is natural in all three variables. This will be noted to be a consequence of the following law:

Lemma 5.1.7 (Exponential Law; Lemma 6.13 in [19])**.** There is an isomorphism of simplicial sets $\Phi \colon Z^{Y \times X} \to \left(Z^Y\right)^X$.

Proof. Let $\phi \colon Y \times X \times \Delta^n \to Z$ be a simplex in $\left(Z^{Y\times X}\right)_n$ be given, then define $\Phi_n(\phi) \colon X \times \Delta^n \to Z^Y$ by letting its value on $x \in X_m$ and $f \in \Delta^n_m$ be given by the map $\Phi_n(\phi)_m(x, f) \colon Y \times \Delta^m \to Z$ defined in degree k as the function

$$\begin{aligned} \Phi_n(\phi)_m(x, f)_k &: Y_k \times \Delta^m_k \to Z_k \\ \Phi_n(\phi)_m(x, f)_k(y, g) &:= \phi_k(y, (X \times \Delta^n)_k(g)(x, f)) \end{aligned}$$

for $y \in Y_k$ and $g \colon [\![k]\!] \to [\![m]\!] \in \Delta^m_k$. Note here, that $(X \times \Delta^n)(g) \colon X_m \times \Delta^n_m \to X_k \times \Delta^n_k$.

Conversely, for a given n-simplex $\psi \colon X \times \Delta^n \to Z^Y$ in $\left(Z^Y\right)^X$, define $\Psi \colon \left(Z^Y\right)^X \to Z^{Y\times X}$ such that $\Psi_n(\psi)$ is given in degree m by

$$\begin{aligned} \Psi_n(\psi)_m &: Y_m \times X_m \times \Delta^n_m \to Z_m \\ \Psi_n(\psi)_m(y, x, f) &:= \psi_m(x, f)(y, [\![m]\!]) \end{aligned}$$

One may check that Φ and Ψ are mutually inverse isomorphisms implying the required result. □

The adjunction stated before (labeled 5.1) is a special case of this, namely there are natural bijections

$$s\mathbf{Set}(X \times Y, Z) \cong \left(Z^{Y\times X}\right)_0 \xrightarrow{\Phi_0} \left(\left(Z^Y\right)^X\right)_0 = s\mathbf{Set}\left(X, Z^Y\right)$$

Lemma 5.1.8 (Theorem 6.9 in [19])**.** If X is a simplicial set and Y is a Kan complex, then the simplicial set Y^X is a Kan complex.

Proof. Omitted. See proof of Theorem 6.9 in [19]. □

[35] Functional simplicial sets are internal homs in the category $s\mathbf{Set}$ and the adjunction $- \times Y \dashv (-)^Y$ makes $s\mathbf{Set}$ a cartesian closed category.

5.2 Homotopy of Simplices

Homotopies between 1-simplices of quasi categories were defined in Definition 2.3.1. For each $n \in \mathbb{N}$, homotopies between n-simplices of Kan complexes may be defined analogously.

Definition 5.2.1 (Homotopic n-simplices; Definition 3.1 in [19]). For a simplicial set X, two n-simplices $e, e' \in X_n$ in X are called **homotopic** (written $e \overset{n}{\sim} e'$) if

(a) $d_i(e) = d_i(e')$ for all $0 \leq i \leq n$.

(b) there exists an $(n+1)$-simplex $\sigma \in X_{n+1}$ such that

   (i) $d_n(\sigma) = e'$ and $d_{n+1}(\sigma) = e$.

   (ii) $d_i(\sigma) = s_{n-1}d_i(e) = s_{n-1}d_i(e')$ for all $0 \leq i < n$.

In such a situation, σ is said to be a **homotopy** from e to e'.

In particular, two 0-simplices are homotopic if and only if they form the endpoints of a single 1-simplex, and for $n = 1$, this definition coincides with Definition 2.3.1 on Kan complexes. Roughly, two n-simplices are homotopic when they form a pair of adjacent faces of an $(n+1)$-simplex whose n other faces are all degeneracies of the non-intersecting faces that form boundaries of the two n-simplices.

Example 5.2.2 ($N(\mathbf{C})$). For $\mathbf{C}$ a small category, as degeneracies in $N(\mathbf{C})$ correspond to inserting identity maps between objects, and k-simplices in $N(\mathbf{C})$ are commutative k-simplices in $\mathbf{C}$, by condition (a) in Definition 5.2, two n-simplices,

$$a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n \quad \text{and} \quad b_0 \xrightarrow{g_1} \cdots \xrightarrow{g_n} b_n$$

of $N(\mathbf{C})$ are homotopic iff they are equal, namely $a_i = b_i$ and $f_i = g_i$ in $\mathbf{C}$, for each index i.

Example 5.2.3 ($Sing(W)$). Let W be a topological space, then two homotopic 1-simplices

$$\gamma_1, \gamma_2 \colon |\Delta^1| \to W$$

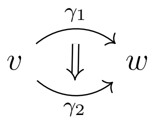
in the singular simplicial set Sing(W), are paths in W with the same start and endpoints (condition (a) in Example 5.2), such that there is a 2-simplex $H \colon |\Delta^2| \to W$ in $Sing(W)_2$ satisfying $d_2(H) = \gamma_1$, $d_1(H) = \gamma_2$ and $d_0(H)$ is constant at the endpoints $d_0(\gamma_1) = d_0(\gamma_2)$ (condition (b)(ii) in Example 5.2). Thus one may construct a homeomorphism $I^2 \to |\Delta^2|$, showing that H defines a path homotopy from γ_1 to γ_2. This proves the claim in Example 2.3.6, remarked previously. In a similar way a homotopy between a pair of 2-simplices of $Sing(W)$ is a homotopy between homotopies, and so on.

Proposition 5.2.4 (Proposition 3.2 in [19]). If X is a Kan complex, for a given $n \geq 0$, the homotopy relation $\overset{n}{\sim}$ is an equivalence relation on X_n.

Proof. This is a variant of the proof of Proposition 2.3.2. The unconvinced reader may seek the proof in [19] as the proof of Proposition 3.2. □

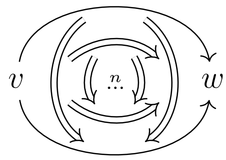
Example 5.2.5 (Fundamental n-groupoid; Example 1.1.1.4 in [16]). For a fixed $n \in \mathbb{N} \cup \{\infty\}$ and a space W, recalling Example 5.2.3, the **fundamental n-groupoid** $\pi_{\leq n}(W)$ is an example of an "n-category" with

- 0-morphisms (or objects) $\text{obj}(\pi_{\leq n}(W)) := Sing(W)_0$, the set of points in W,
- 1-morphisms (or morphisms) $\pi_{\leq n}(W)\,(v, w)_1 := \{\gamma \in Sing(W)_1 \mid \gamma \text{ is a homotopy from } v \text{ to } w\}$, i.e. the set of paths $v \to w$ from v to w in W,

- 2-morphisms $\pi_{\leq n}(W)\,(\gamma_1, \gamma_2)_2 := \{\Gamma \in Sing(W)_2 \mid \Gamma$ is a homotopy from γ_1 to $\gamma_2\}$, i.e. the path homotopies 

 between paths $\gamma_1, \gamma_2 \colon v \to w$ in W,

  ⋮

- n-morphisms as homotopies 

 between $(n-1)$-morphisms,[36]

modulo the equivalence relation where two n-morphisms are equivalent iff there is an $(n+1)$-morphism between them (here for $n = \infty$, this relation vanishes). As homotopies and paths are always reversible, it follows that for this category, r-morphisms are invertible for every $r \geq 0$. This is often incorporated for in the terminology, by calling it an $(n, 0)$**-category** or an n**-groupoid**.

In particular, when $n = 0$, the fundamental 0-groupoid $\pi_{\leq 0}(W) = \pi_0(W)$ is the set (or 0-category) of connected components of W, and when $n = 1$, the fundamental (1-)groupoid $\pi_{\leq 1}(W)$ is the category (or 1-category) from Example 2.3.6, and the notation is consistent.

Definition 5.2.6 (Relative Homotopy; Definition 3.3 in [19])**.** For a simplicial set X, two n-simplices $e, e' \in X_n$ in X are called **homotopic relative** to a subsimplicial set A of X (written $e \overset{n}{\sim} e'$ rel A) if

(a) $d_i(e) = d_i(e')$ for all $1 \leq i \leq n$ and there is a homotopy $d_0(e) \overset{n-1}{\sim} d_0(e')$ via a simplex $\tau \in A_n$.

(b) there exists an $(n+1)$-simplex $\sigma \in X_{n+1}$ such that

  (i) $d_0(\sigma) = \tau$, $d_n(\sigma) = e'$ and $d_{n+1}(\sigma) = e$.

  (ii) $d_i(\sigma) = s_{n-1}d_i(e) = s_{n-1}d_i(e')$ for all $1 \leq i < n$.

In such a situation σ is said to be a **relative homotopy** from e to e' relative to A.

In rough terms, this is the same as the definition of a homotopy between n-simplices, except, the face opposite the 0^{th} vertex in the relative homotopy σ (with notation as above) is a degenerate n-simplex that gives a homotopy τ in A, between the two faces of e and e' that it intersects.

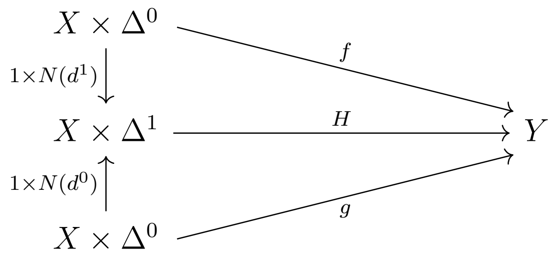
Proposition 5.2.7 (Proposition 3.4 in [19])**.** If (X, A) is a Kan pair, for $n \geq 1$, the relation $\overset{n}{\sim}$ rel A is an equivalence relation on the set of n-simplices $x \in X_n$ satisfying $d_0(x) \in A_{n-1}$.

Proof. Omitted. See the proof of Proposition 3.4 in [19]. □

5.3 Homotopy of Simplicial Maps

Definition 5.3.1 (Homotopy of Maps; Ch. I, §6, Introduction in [9])**.** Two given simplicial maps $f, g \colon X \to Y$ are said to be (simplicially) **homotopic** (written $f \sim g$) if there is a simplicial map $H \colon X \times \Delta^1 \to Y$ such that the diagram

[36]Here the symbol $\overset{n}{\therefore}$ in the schematic, is used to indicate that the suggested process terminates after n iterates.



in s**Set** commutes. In this situation H is called a **homotopy** from f to g. Equivalently, a homotopy from f to g is a 1-simplex H in $\left(Y^X\right)_1$ such that $d_1(H) = f$ and $d_0(H) = g$.[37]

Proposition 5.3.2 (Corollary 6.12 in [19])**.** For X a simplicial set and Y a Kan complex, homotopy of simplicial maps is an equivalence relation on the set $s\mathbf{Set}(X, Y)$.

Proof. (Alternate Proof) Let simplicial maps $f, g, h \in s\mathbf{Set}(X, Y)$ be given.

(Reflexivity) Then f is homotopic to f via the homotopy $Y^X(s^0)(f) = f \circ (1 \times N(s^0)) \colon X \times \Delta^1 \to Y$ as the relevant diagram commutes by the cosimplicial identities and the functoriality of the nerve functor.

(Symmetry) Now suppose $H \colon X \times \Delta^1 \to Y$ is a homotopy from f to g. Then the pair of 1-simplices $(-, H, f \circ (1 \times N(s^0)))$ is compatible as $d_1(f \circ (1 \times N(s^0))) = f = d_1(H)$, so consider the horn

$$\Lambda^2_0 \xrightarrow{\left(-, H, f \circ (1 \times N(s^0))\right)} X^Y$$

in the functional simplicial set X^Y. As Y is a Kan complex, by Lemma 5.1.8 there is an extension that represents a simplex $\sigma \in \left(Y^X\right)_2$. Then $d_0(\sigma)$ is a homotopy from g to f as by the simplicial identities, $d_0(d_0(\sigma)) = d_0 d_1(\sigma) = d_0(H) = g$ and $d_1(d_0(\sigma)) = d_0 d_2(\sigma) = d_0(f \circ (1 \times N(s^0))) = f$.

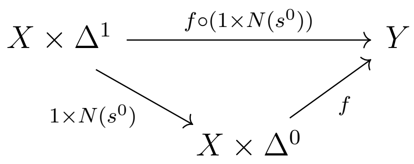
(Transitivity) Suppose f is homotopic to g via H_1 and g is homotopic to h via H_2, then the pair $(H_2, -, H_1)$ is compatible, so consider the horn

$$\Lambda^2_1 \xrightarrow{(H_2, -, H_1)} X^Y$$

in the simplicial set X^Y. Again using Lemma 5.1.8 there is an extension representing a simplex $\tau \in \left(Y^X\right)_2$ with its face $d_1(\tau)$ satisfying by the simplicial identities, $d_0(d_1(\tau)) = d_0 d_0(\tau) = d_0(H_2) = h$ and $d_1(d_1(\tau)) = d_1 d_2(\tau) = d_1(H_1) = f$. Thus $d_1(\tau)$ is a homotopy from f to h as required. □

In fact there is a category $Ho(\mathbf{Kan})$ called the **homotopy category** of Kan complexes, with objects $\mathrm{obj}(Ho(\mathbf{Kan})) := \mathrm{obj}(\mathbf{Kan})$ and morphisms $Ho(\mathbf{Kan})(X, Y) := s\mathbf{Set}(X, Y)/\sim$.[38]

Remark 5.3.3. Implicitly used in the proof of Proposition 5.3.2, for a simplicial map $f \colon X \to Y$ the **constant homotopy** from f to itself is the degenerate 1-simplex $Y^X(s^0)(f) = f \circ (1 \times N(s^0))$, thus the homotopy $X \times \Delta^1 \to Y$ which factors



through the projection $1 \times N(s^0)$ as f.

[37] With this noted, degenerate 1-simplices in X^Y will correspond to constant homotopies, and thus a homotopy between a pair of homotopies in $H_1, H_2 \in \left(Y^X\right)_1$ from f to g, is then a homotopy between these 1-simplices.

[38] This should not be confused with the notation $Ho(X_\bullet)$ used in Definition 2.3.4 for the homotopy category of a *particular* simplicial set $X_\bullet$.

Proposition 5.3.4 (Corollary 14.5 in [19])**.** A homotopy $H\colon X \times \Delta^1 \to Y$ of simplicial maps between simplicial sets X and Y induces a (topological) homotopy $\overline{H}\colon |X| \times I \to |Y|$ of continuous maps.

Proof. If $H\colon X \times \Delta^1 \to Y$ is a homotopy from f to g, then as $|X| \times I$ is a CW complex, by Theorem 4.4.6 and Example 4.3.2, there is a homeomorphism $\varphi\colon |X| \times I \to |X \times \Delta^1|$ implying the result. In particular, $\overline{H}$ is the composite $H \circ \varphi$. □

Example 5.3.5 (Natural Transformation)**.** If $F, G\colon \mathbf{C} \to \mathbf{D}$ are functors between small categories $\mathbf{C}$ and $\mathbf{D}$ such that there is a natural transformation $\eta\colon F \implies G$, then η defines a homotopy between simplicial maps $N(F)$ and $N(G)$. In particular, the simplicial map determined by the function

$$\begin{aligned} N(\mathbf{C})_1 \times \Delta^1_1 &\to N(\mathbf{D})_1 \\ \left(f, 1_{[\![1]\!]}\right) &\mapsto N\eta(f) \end{aligned}$$

makes the relevant diagram commute. Moreover, due to Proposition 5.3.4, there is a homotopy between topological spaces $B\mathbf{C}$ and $B\mathbf{D}$.

Remark 5.3.6. As the nerve functor preserves limits, the homotopy $N(\mathbf{C}) \times \Delta^1 \to N(\mathbf{D})$ can be interpreted as a simplicial map $N(\mathbf{C} \times [\![1]\!]) \to N(\mathbf{D})$, thus applying the functor h to this simplicial map gives by the natural isomorphism $hN \implies 1_{\mathbf{Cat}}$ provided by the counit, an alternate definition for a natural transformation from F to G between small categories, as a functor $\mathbf{C} \times [\![1]\!] \to \mathbf{D}$ mapping $(-, \{0\})$ to F and $(-, \{1\})$ to G. This can of course be seen directly as well.

Definition 5.3.7 (Relative Homotopy of Maps; Definition 8.13 in [7])**.** A simplicial homotopy $H\colon X \times \Delta^1 \to Y$ from a simplicial map f to a simplicial map g is called a (simplicial) **homotopy relative** to a subsimplicial set A if A is a subsimplicial set of X such that $f|_A = g|_A$ and $H|_{A\times\Delta^1}\colon A \times \Delta^1 \to Y$ is the constant homotopy. In this situation f and g are called **homotopic relative** to A (written $f \sim g$ rel A).

Theorem 5.3.8 (Lemma 9.5 in [7])**.** Let X be a Kan complex and $x, y \in X_n$ be two n-simplices in X, then x is homotopic to y if and only if the corresponding maps in $s\mathbf{Set}(\Delta^n, X)$ (as per Lemma 2.1.4) for x and y, are homotopic relative to $\partial\Delta^n$.

Proof. Omitted. See Lemma 9.5 in [7]. □

5.4 Simplicial Homotopy Groups

Simplicial homotopy groups are combinatorial analogs of topological homotopy groups. In fact one may define the simplicial homotopy group of a pointed simplicial set $(X, \langle x\rangle)$ to be the usual (topological) homotopy group of the pointed space $(|X|, |\langle x\rangle|)$ obtained by taking its (pairwise) realisation. However, simplicial homotopy groups may be described purely combinatorially for pointed Kan complexes, without referring to topological spaces.

Notation 5.4.1. Sometimes the symbol $[z]$ (or in some cases $\overline{z}$) may refer to a certain equivalence class generated by an element z. However, when it is said that z is a representative of $[z]$, the generator is neglected and the symbol $[z]$ is treated as an arbitrary symbol (independent of z).[39]

[39] The philosophy behind this choice of notation is that once the equivalence class has been generated, the generator is "forgotten".

5.4.1 Absolute Version

Definition 5.4.2 (Path Components; Definition 3.6 in [19])**.** The set of **path components** of a pointed Kan complex $(X, \langle x \rangle)$ is defined to be the set

$$\pi_0(X, \langle x \rangle) := X_0/ \overset{0}{\sim} .$$

(Definitions 3.6 and 4.1 in [19]) For a fixed $n \geq 1$, and $(X, \langle x \rangle)$ a pointed Kan complex, based at a basepoint $\langle x \rangle$ generated by a 0-simplex x of X, define the set

$$\pi_n(X, \langle x \rangle) := \{y \in X_n \mid d_i(y) = \langle x \rangle_{n-1} \text{ for all } i\}/ \overset{n}{\sim}$$

where $\langle x \rangle_{n-1}$ is the singleton obtained by (degeneracies of) x. For $[y], [z] \in \pi_n(X, \langle x \rangle)$ and for some representatives y of $[y]$ and z of $[z]$, then the $(n+1)$-tuple $(\langle x \rangle_n, \dots, \langle x \rangle_n, y, -, z)$ is compatible by definition of the set $\pi_n(X, \langle x \rangle)$ and by condition (a) in Definition 5.2.1, so there is a horn

$$\Lambda^{n+1}_n \xrightarrow{(\langle x \rangle_n, \dots, \langle x \rangle_n, y, -, z)} X$$

in X. Using the Kan condition, let α be a filler of this horn.

Lemma 5.4.3 (Lemma 4.2 in [19])**.** For $n \geq 1$, with the same notations as above and $[d_n(\alpha)]$, the equivalence class generated by $d_n(\alpha)$ (see Notation 5.4.1), the assignment $([y], [z]) \mapsto [y]\cdot[z] := [d_n(\alpha)]$ defines a well defined binary operation

$$\cdot \colon \pi_n(X, \langle x \rangle) \times \pi_n(X, \langle x \rangle) \to \pi_n(X, \langle x \rangle).$$

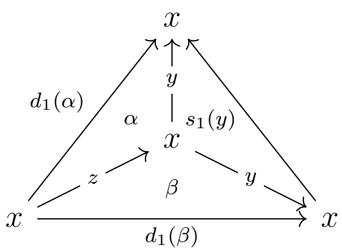
Proof. Suppose β is another $(n+1)$-simplex that is a filler of the horn

$$\Lambda^{n+1}_n \xrightarrow{(\langle x \rangle_n, \dots, \langle x \rangle_n, y, -, z)} X$$

in X. Then $d_{n+1}(\beta) = z = d_{n+1}(\alpha)$ and by the simplicial identities, $d_{n-1}(\alpha) = d_{n-1}(\beta) = y = d_{n+1}s_n(y)$ so there is a $(n+2, n)^{th}$ horn

$$\Lambda^{n+2}_n \xrightarrow{(\langle x \rangle_{n+1}, \dots, \langle x \rangle_{n+1}, s_n(y), -, \alpha, \beta)} X$$

in X, then by the Kan condition there is a filler $\Sigma \in X_{n+2}$, but then by the simplicial identities, the $(n+1)$-simplex $d_n(\Sigma)$ satisfies $d_n(d_n(\Sigma)) = d_n d_{n+1}(\Sigma) = d_n(\alpha)$; $d_{n+1}d_n(\Sigma) = d_n(d_{n+2}(\Sigma)) = d_n(\beta)$; $d_{n-1}(d_n(\Sigma)) = d_{n-1}d_{n-1}(\Sigma) = d_{n-1}s_n(y) = s_{n-1}(d_{n-1}(y)) = s_{n-1}(d_{n-1}(d_{n-1}(\alpha))) = s_{n-1}(d_{n-1}d_n(\alpha))$, whence, $d_{n-1}d_{n-1}(\Sigma) = d_{n-1}s_n(y) = s_{n-1}(d_{n-1}(y)) = s_{n-1}(d_{n-1}(d_{n-1}(\beta)))$; and for $i < n-1$, $d_i(d_n(\Sigma)) = d_{n-1}d_i(\Sigma) = \langle x \rangle_n = s_{n-1}d_{n-1}d_i(\alpha) = s_{n-1}d_i(d_n(\alpha)) = s_{n-1}d_i(d_n(\beta))$. Thus $d_n(\Sigma)$ is a homotopy between $d_n(\alpha)$ and $d_n(\beta)$. For the case when $n = 1$, the 3-simplex Σ is depicted below.



Now in order to show that the product $[y] \cdot [z]$ is independent of the choice of representatives of $[y]$ and $[z]$, let z and z' be representatives of $[z]$, then there is a homotopy of simplices $z \overset{n}{\sim} z'$ via an $(n+1)$-simplex γ from z to z'. Let δ be a filler of the horn

$$\Lambda^{n+1}_n \xrightarrow{(\langle x \rangle_n, \ldots, \langle x \rangle_n, y, -, z')} X$$

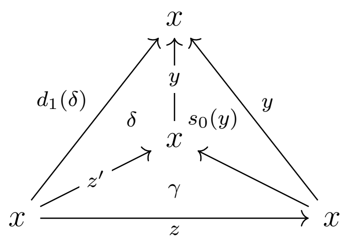
in X, then $d_n(\gamma) = z' = d_{n+1}(\delta)$ and by the simplicial identities, $d_{n-1}(\gamma) = s_{n-1}d_{n-1}(z) = \langle x \rangle_n = s_{n-1}d_n(y) = d_{n+1}s_{n-1}(y)$ and $d_{n-1}(\delta) = y = d_{n-1}s_{n-1}(y)$, so there is a horn

$$\Lambda^{n+2}_{n+1} \xrightarrow{(\langle x \rangle_{n+1}, \ldots, \langle x \rangle_{n+1}, s_{n-1}y, \delta, -, \gamma)} X$$

in X which admits a filler $\Pi \in X_{n+2}$ by the Kan condition. By the simplicial identities, the $(n+1)$-simplex $d_{n+1}(\Pi)$ satisfies $d_n(d_{n+1}(\Pi)) = d_n d_n(\Pi) = d_n(\delta)$; $d_{n+1}(d_{n+1}(\Pi)) = d_{n+1}d_{n+2}(\Pi) = d_{n+1}(\gamma) = z'$; $d_{n-1}(d_{n+1}(\Pi)) = d_n d_{n-1}(\Pi) = d_n(s_{n-1}y) = y$. Thus

$$[y] \cdot [z] = [d_n(d_{n+1}(\Pi)) = d_n d_n(\Pi)] = [y] \cdot [z'].$$

In particular, when $n = 1$, the 3-simplex Π is depicted below.

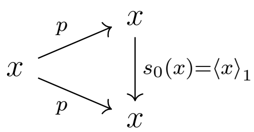


The argument for the fact that the first component is independent of the choice of representative, is similar. □

Theorem 5.4.4 (Lemma 4.3 in [19]). For $n \geq 1$, the set $\pi_n(X, \langle x \rangle)$ equipped with the operation $\cdot$ defined in Lemma 5.4.3 is a group called the n^{th} (simplicial) **homotopy group** of the pointed Kan complex $(X, \langle x \rangle)$.

Sketch of Proof. Let $[p], [q], [r] \in \pi_n(X, \langle x \rangle)$ be given and p, q and r be representatives of $[p], [q]$ and $[r]$, respectively.

(Identity) The identity element is the class of the element $\langle x \rangle_n \in X_n$. By the simplicial identities, $s_n(p)$ is an $(n+1)$-simplex with $d_{n+1}(s_n(p)) = d_n(s_n(p)) = p$ and $d_i(s_n(p)) = \langle x \rangle_n$ for each $i < n$. In the one dimensional case, $s_1(p)$ is the 2-simplex depicted below.



(Inevitability) The inverse $[p]^{-1}$ of the class $[p]$ is obtained as the class of the $(n+1)^{th}$ face of a filler of the horn

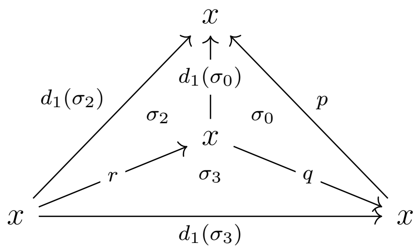
$$\Lambda^{n+1}_{n+1} \xrightarrow{(\langle x \rangle_n, \ldots, \langle x \rangle_n, p, \langle x \rangle_n, -)} X$$

in X. It follows by construction that $[p] \cdot [p]^{-1}$ is the class of the identity $[\langle x \rangle_n]$.

(Associativity) By the Kan condition pick $(n+1)$-simplices $\sigma_{n-1}, \sigma_{n+1}$ and σ_{n+2} satisfying $d_i(\sigma_j) = \langle x \rangle_n$ for all $i < n-1$, and for all $j \in \{n-1, n+1, n+2\}$, and the following equations

$$p = d_{n-1}(\sigma_{n-1}); \quad \begin{aligned} q &= d_{n-1}(\sigma_{n+2}) \\ &= d_{n+1}(\sigma_{n-1}); \end{aligned} \quad \begin{aligned} r &= d_{n+1}(\sigma_{n+2}) \\ &= d_{n+1}(\sigma_{n+1}); \end{aligned} \quad d_{n-1}(\sigma_{n+2}) = d_n(\sigma_{n-1}).$$

Using the Kan condition, pick a simplex $\Sigma \in X_{n+2}$ such that $d_i(\Sigma) = \sigma_i$, $i \in \{n-1, n, n+1\}$, and $d_i(\Sigma) = \langle x \rangle$ if $i < n-1$. In particular, for the case when $n = 1$ the 3-simplex Σ is depicted below.



By simplex chasing,[40]

$$([p] \cdot [q]) \cdot [r] = [d_n(\sigma_{n-1})] \cdot [r] = [d_n(\sigma_{n+1}) = d_n(d_n(\Sigma))] = [p] \cdot [d_n(\sigma_{n+2})] = [p] \cdot ([q] \cdot [r])$$

and so the operation $\cdot$ on $\pi_n(X, \langle x \rangle)$ defines a group. □

Proposition 5.4.5 (Proposition 4.4 in [19])**.** For $n \geq 2$, the group $\pi_n(X, \langle x \rangle)$ is abelian.

Proof. Omitted. See the proof of Proposition 4.4 in [19]. □

Example 5.4.6 ($\pi_n(\mathbf{B}G_\bullet, \langle \star \rangle)$)**.** For the nerve $\mathbf{B}G_\bullet$ of the groupoid $\mathbf{B}G$ corresponding to a group G (which is a Kan complex by Proposition 2.2.10), based at the unique 0-simplex $\star$ of $\mathbf{B}G_\bullet$, using Example 2.1.2, the set

$$\pi_n(\mathbf{B}G_\bullet, \langle \star \rangle) = \{(g_1, \ldots, g_n) \in \mathbf{B}G_n \cong G^{\times n} \mid d_i(g_1, \ldots, g_n) = \langle \star \rangle_{n-1} \text{ for each } i\} / \overset{n}{\sim}$$

is the singleton $\{(e, \ldots, e)\}$ consisting of the degeneracy of $\star$ in degree n, for each $n \geq 2$, and is the singleton $\{\star\}$ when $n = 0$. For $n = 1$, note that by definition of the face map on $\mathbf{B}G_1$, $d_0(g_1) = d_1(g_1) = \star$. Also note that by Example 5.2.2, two 1-simplices in $\mathbf{B}G_1$ are homotopic if and only if they are equal. Now for elements $g, h \in \mathbf{B}G_1 \cong G$ a horn

$$\Lambda^2_1 \xrightarrow{(g,-,h)} \mathbf{B}G_1$$

has a filler $(g, h) \in \mathbf{B}G_2 \cong G^{\times 2}$ and $d_1(g, h) = gh$, so there is an isomorphism

$$\pi_n(\mathbf{B}G_\bullet, \langle \star \rangle) \cong \begin{cases} G, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

A (pointed) Kan complex $(X, \langle x \rangle)$, with only a single non-trivial simplicial homotopy group $G \cong \pi_n(X, \langle x \rangle)$ is called an **Eilenberg-MacLane** simplicial set of type (G, n).

Example 5.4.7 ($\pi_n(Sing(W), \langle w \rangle)$)**.** Recall that $Sing(W)$ is a Kan complex by Example 2.2.7. Let W be a topological space, then $w \in Sing(W)_0$ is a function $|\Delta^0| \to W$ and hence just a point $\dot{w}$ in W. Then the simplicial homotopy group

$$\begin{aligned} \pi_n(Sing(W), \langle w \rangle) &= \{y \colon |\Delta^n| \to W \mid \langle w \rangle_{n-1} = d_i(y) \colon |\Delta^{n-1}| \to W \text{ for all } i\} / \overset{n}{\sim} \\ &= \{y \colon |\Delta^n| \to W \mid \partial |\Delta^n| \to W \text{ is the constant map at } \dot{w}\} / \overset{n}{\sim} \end{aligned}$$

is by Example 5.2.3, the set $\pi_n(W, \dot{w})$. The product operation $\cdot$ on the simplicial homotopy group $\pi_n(Sing(W), \langle w \rangle)$ is an alternate definition for the product in the topological homotopy group $\pi_n(W, \dot{w})$. For instance, for the case when $n = 1$, the product $\cdot$ on $\pi_1(Sing(W), \langle w \rangle)$ is seen to be the juxtaposition of (topologocal) homotopy classes of loops in W, based at the point $\dot{w} \in W$.

[40] More formally, this would have to be shown by the simplicial identities.

5.4.2 Relative Version

(Definitions 3.6 and 4.5 in [19]) For an integer $n \geq 2$, and $(X, A, \langle x \rangle)$ a pointed Kan pair, based at a basepoint $\langle x \rangle$ generated by a 0-simplex x of X, define the set

$$\pi_n(X, A, \langle x \rangle) := \{y \in X_n \mid d_i(y) = \langle x \rangle_{n-1} \text{ for all } i \neq 0 \text{ and } d_0(x) \in A_{n-1}\} / \left(\overset{n}{\sim} \text{ rel } A \right)$$

and as before, $\langle x \rangle_{n-1}$ is the singleton obtained by (degeneracies of) x in degree $n-1$. For classes $[y], [z] \in \pi_n(X, \langle x \rangle)$[41] and for some representatives y of $[y]$ and z of $[z]$, by the simplicial identities, the compatibility condition for the horn

$$\Lambda^n_{n-1} \xrightarrow{\left(\langle x \rangle_{n-1}, \ldots, \langle x \rangle_{n-1}, d_0(y), -, d_0(z)\right)} A$$

in A, is satisfied. By the Kan condition, there is a filler α satisfying $d_{n-2}(\alpha) = d_0(y)$ and $d_n(\alpha) = d_0(z)$. Thus there is a horn

$$\Lambda^{n+1}_n \xrightarrow{\left(\alpha, \langle x \rangle_n, \ldots, \langle x \rangle_n, y, -, z\right)} X$$

in X which admits a filler β.

Lemma 5.4.8 (Relative Homotopy Group; Lemma 4.6 in [19])**.** With the same notations as above, the assignment $([y], [z]) \mapsto [y] \cdot [z] := [d_n(\beta)]$ defines a well defined binary operation

$$\cdot \colon \pi_n(X, A, \langle x \rangle) \times \pi_n(X, A, \langle x \rangle) \to \pi_n(X, A, \langle x \rangle).$$

Proof. Omitted. Similar to the absolute case. □

Theorem 5.4.9 (Lemma 4.6 in [19])**.** For $n \geq 2$, the set $\pi_n(X, A, \langle x \rangle)$ equipped with the operation $\cdot$ in Lemma 5.4.8 is a group called the n^{th} (simplicial) **relative homotopy group** of the pointed Kan pair $(X, A, \langle x \rangle)$.

Proof. Omitted. Similar to the absolute case. □

Note that in particular, for $(X, \langle x \rangle)$ a pointed Kan complex, then $\pi_n(X, \langle x \rangle, \langle x \rangle) = \pi_n(X, \langle x \rangle)$.

Proposition 5.4.10 (Proposition 4.8 in [19])**.** For $n \geq 3$, the group $\pi_n(X, A, \langle x \rangle)$ is abelian.

Proof. Omitted. Similar to the proof of Proposition 4.4 in [19]. □

It may be easily verified that π_n defines a functor from the category of pointed Kan complexes (respectively of pointed Kan pairs) to the category of groups for $n \geq 1$ (respectively $n \geq 2$), defined for the absolute version, as the functor carrying a simplicial map of pairs $f \colon (X, \langle x \rangle) \to (Y, \langle f(x) \rangle)$ between pointed Kan complexes, to the homomorphism $\pi_n(f) \colon \pi_n(X, \langle x \rangle) \to \pi_n(Y, \langle f(x) \rangle)$ defined by $\pi_n(f)([y]) := [f(y)]$[42] for each $[y] \in \pi_n(X, \langle x \rangle)$ and some chose representative y of $[y]$. Natural transformations relating these functors will be provided by the following lemma:

Lemma 5.4.11 (Theorem 3.7 in [19])**.** Let $\partial^\pi_{n+1} \colon \pi_{n+1}(X, A, \langle x \rangle) \to \pi_n(X, \langle x \rangle)$ be the map that carries an element $[y] \in \pi_{n+1}(X, A, \langle x \rangle)$ with some representative y, to the class $[d_0(y)] \in \pi_n(A, \langle x \rangle)$.[43] For a pointed Kan complex $(X, A, \langle x \rangle)$, there is a long exact sequence

[41] Note here that for brevity, the same notations indicating homotopy classes as in the absolute versions, have been used. The hope is that the distinction is clear from the context.

[42] Note that the square bracket $[y]$ denotes homotopy classes in X whilst $[f(y)]$ denotes homotopy classes in A.

[43] This is intuitively thought of as the map induced by applying d_0 level-wise, however, the assignment applying d_0 level-wise does *not* define a simplicial map.

$$\cdots \xrightarrow{\partial^\pi_{n+2}} \pi_{n+1}(A,\langle x\rangle) \xrightarrow{\pi_{n+1}(i)} \pi_{n+1}(X,\langle x\rangle) \xrightarrow{\pi_{n+1}(j)} \pi_{n+1}(X,A,\langle x\rangle) \xrightarrow{\partial^\pi_{n+1}} \pi_n(A,\langle x\rangle) \xrightarrow{\pi_n(i)} \pi_n(X,\langle x\rangle) \xrightarrow{\pi_n(j)} \pi_n(X,A,\langle x\rangle) \xrightarrow{\partial^\pi_n} \cdots$$

induced by inclusions i and j.

Proof. Only exactness at $\pi_n(A,\langle x\rangle)$ is shown, the rest may be found in [19] as the proof of Lemma 3.7.

[$\operatorname{img}(\partial^\pi_{n+1}) \subseteq \ker(\pi_n(i))$] Let $[y] \in \pi_{n+1}(X,A,\langle x\rangle)$, then $\partial^\pi_{n+1}([y]) = [d_0(y)] \in \operatorname{img}(\partial^\pi_{n+1})$. It is required to show that $[d_0(y)] = [\langle x\rangle_n]$, the identity *in* $\pi_n(X,\langle x\rangle)$. Let y be a representative for $[y]$, then consider the horn

$$\Lambda^{n+2}_{n+2} \xrightarrow{(\langle x\rangle_{n+1},\ldots,\langle x\rangle_{n+1},s_{n+1}d_0(y),-)} X$$

in X.[44] By the Kan condition, there is a filler, say $z \in X_{n+2}$ which satisfies $d_i(z) = \langle x\rangle$ for $i \leq n$, $d_{n+1}(z) = s_{n+1}d_0(z)$ whilst the face $d_{n+2}(z)$, satisfies by the simplicial identities, for all $i \leq n$, $d_i(d_{n+2}(z)) = d_{n+1}(d_i(z)) = d_{n+1}(\langle x\rangle_{n+1}) = \langle x\rangle_n$ and $d_{n+1}(d_{n+2}(z)) = d_{n+1}(d_{n+1}(z)) = d_{n+1}(s_{n+1}d_0(z)) = d_0(z)$. Thus $d_0(y) \overset{n}{\sim} \langle x\rangle_n$ (in X_n).

[$\operatorname{img}(\partial^\pi_{n+1}) \supseteq \ker(\pi_n(i))$] Let $[y] \in \ker(\pi_n(i)) \subseteq \pi_n(A,\langle x\rangle)$, then $y \overset{n}{\sim} \langle x\rangle_n$ *in* X_n, so say $\sigma \in X_{n+1}$ is a homotopy from y to $\langle x\rangle_n$, then $d_i(\sigma) = \langle x\rangle_n$ for each $i \leq n$, and $d_{n+1}(\sigma) = y$. Now consider the horn

$$\Lambda^{n+2}_{n+2} \xrightarrow{(\sigma,\langle x\rangle_{n+1},\ldots,\langle x\rangle_{n+1},-)} X$$

in X. Then by the Kan condition, there is an $(n+2)$-simplex $\Sigma \in X_{n+2}$ for which, by the simplicial identities, $d_0(d_{n+2}(\Sigma)) = d_{n+1}(d_0(\Sigma)) = d_{n+1}(\sigma) = y$. In other words, $\partial^\pi_{n+1}([d_{n+2}(\Sigma)]) = [y]$, so $[y] \in \operatorname{img}(\partial^\pi_{n+1})$. □

5.5 Fibrations

Analogous to fibration sequences in the category of topological spaces, fibration sequences may be defined in the category s**Set** of simplicial sets, and will be seen to induce long exact sequences of simplicial homotopy groups.

Definition 5.5.1 (Kan Fibration; Ch. I, §3 after Corollary 3.2 in [9])**.** A simplicial map $p\colon E \to B$ is called a **Kan fibration** if any given horn Λ^n_k and arrows fitting into a commutative diagram of solid arrows

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & B \end{array}$$

admits a simplicial map expressed by the dashed arrow, rendering the entire diagram commutative. The triple (E,p,B) is called a **fibre space**, E is called the **total space**[45] and B is called the **base space**. Note that when B is the terminal simplicial set Δ^0, then the unique map $p\colon E \to \Delta^0$ is a Kan fibration if and only if every horn in E has a filler, that is E is a Kan complex.

Remark 5.5.2 (Combinatorial Definition; Definition 7.1 in [19])**.** Equivalently, p is a Kan fibration if all n-tuples of $(n-1)$-simplices $(x_0,\ldots,x_{k-1},-,x_{k+1},\ldots,x_n)$ in E that sayisfy the condition $d_i x_j = d_{j-1}x_i$ for all $k \neq i < j \neq k$ and for all n-simplices $\sigma \in B_n$ satisfying $d_i(\sigma) = p(x_i)$ for $i \neq k$, there is an n-simplex $x \in E_n$ such that $d_i(x) = x_i$ for each $i \neq k$ and $p(x) = \sigma$.

[44]The compatibility condition is easily verified here.

[45]The letter E used to denote the total space is presumed to have originated from the French word “ensemble”.

Example 5.5.3 (Map induced by a Serre Fibration; pg. 10 in [9])**.** Let $incl_0 \colon \{0\} \to I$ be the inclusion into the point $\{0\}$ and let $f \colon V \to W$ be a continuous map between topological spaces V and W such that for all n, and all arrows fitting into a commutative diagrams of solid arrows

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & V \\ {\scriptstyle 1\times incl_0}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ D^n \times I & \longrightarrow & W \end{array}$$

in **Top**, there is a dashed arrow making the whole diagram commute. In this case f is called a **Serre fibration**. The map $Sing(f) \colon Sing(V) \to Sing(W)$ induced by f is a Kan fibration as a given commutative diagram of solid arrows

$$\begin{array}{ccc} \Lambda^{n+1}_k & \longrightarrow & Sing(V) \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle Sing(f)} \\ \Delta^{n+1} & \longrightarrow & Sing(W) \end{array}$$

where i is the inclusion, requiring a dashed arrow so that the resulting diagram commutes, is equivalent via the adjunction $|\cdot| \dashv Sing$, to requiring a dashed arrow making the diagram

$$\begin{array}{ccc} |\Lambda^{n+1}_k| & \longrightarrow & V \\ {\scriptstyle |i|}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ |\Delta^{n+1}| & \longrightarrow & W \end{array}$$

commute. Whence, the map $|i|$ induced by the inclusion, is up to compositions with homeomorphisms, the inclusion $D^n \times \{0\} \xhookrightarrow{1\times incl_0} D^n \times I$, implying the claim as f is a Serre fibration.

In fact the argument in Example 5.5.3 shows additionally that a map of simplicial sets in a Kan fibration *only if* it is a Serre fibration.

Definition 5.5.4 (Fibre Sequence; Definition 7.1 and Remark 7.2 in [19])**.** For (E, p, B), a fibre space and a vertex $b \in B_0$, the **fibre** of p over $\langle b \rangle$ is the preimage $F := p^{-1}(\langle b \rangle)$.[46] For a vertex $e \in F_0$, the sequence of simplicial pairs

$$(F, \langle e \rangle) \xhookrightarrow{i} (E, \langle e \rangle) \xrightarrow{p} (B, \langle b \rangle)$$

where i is the inclusion, is called a **fibre sequence**.

Lemma 5.5.5 (Propositions 7.3 and 7.5 in [19])**.** If (E, p, B) is a fibre space and F is the fibre of p over some basepoint $\langle b \rangle$ of B, then

(i) F is a Kan complex.

(ii) if p is surjective and E is Kan complex, then B is a Kan complex.

(iii) if B is a Kan complex, then E is a Kan complex.

Proof. (i) Let a horn

$$\Lambda^n_k \xrightarrow{f} F \tag{5.2}$$

in F be given. Consider this map to a map into E by post-composing with the inclusion $i \colon F \hookrightarrow E$ and set $c_{\langle b \rangle} \colon \Delta^n \to B$ to be the unique map to the basepoint $\langle b \rangle$ in B to obtain the commutative diagram

[46] One may check here that the fibre F is indeed a simplicial set, it will be seen that it is also a Kan complex.

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{i\circ f} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \xrightarrow[c_{\langle b\rangle}]{} & B \end{array}$$

of solid arrows for which there exists a dashed arrow making the whole diagram commute as p is a Kan fibration. Since $p^{-1}(\langle b\rangle) = F$, by commutativity of the diagram, the dashed arrow may be considered to be a map $\Delta^n \to F$ working as the required filler horn labelled (5.2). Thus F is a Kan complex.

(ii) Omitted. See the proof of Proposition 7.5 in [19] for a proof using Remark 5.5.2.

(iii) Let B be a Kan complex and let $f\colon \Lambda^k_n \to E$ be a given horn in E, then there is a horn

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{p\circ f} & B \\ \big\downarrow & \nearrow_{g} & \\ \Delta^n & & \end{array}$$

in B that admits a filler expressed by the dashed arrow g. In particular, this gives a diagram

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \xrightarrow[g]{} & B \end{array}$$

of solid arrows that admits a dashed arrow making both triangles commute becuase p is a Kan fibration. Showing in particular that E is a Kan complex. □

Theorem 5.5.6 (Long Exact Sequence of Simplicial Homotopy Groups; Theorem 7.6 in [19])**.** If $(F, \langle e\rangle) \overset{i}{\hookrightarrow} (E, \langle e\rangle) \overset{p}{\to} (B, \langle b\rangle)$ is a fibre sequence where $(F, \langle e\rangle)$, $(E, \langle e\rangle)$ and $(B, \langle b\rangle)$ are all pointed Kan complexes, then the induced maps $\pi_n(E, F, \langle e\rangle) \to \pi_n(B, \langle b\rangle)$ are isomorphisms for each $n \geq 1$ and the induced sequence

$$\cdots \xrightarrow{\partial^\#_{n+2}} \pi_{n+1}(F, \langle e\rangle) \xrightarrow{\pi_{n+1}(i)} \pi_{n+1}(E, \langle e\rangle) \xrightarrow{\pi_{n+1}(p)} \pi_{n+1}(B, \langle b\rangle) \xrightarrow{\partial^\#_{n+1}} \pi_n(F, \langle e\rangle) \xrightarrow{\pi_n(i)} \pi_n(E, \langle e\rangle) \xrightarrow{\pi_n(p)} \pi_n(B, \langle b\rangle) \xrightarrow{\partial^\#_n} \cdots$$

of simplicial homotopy groups, is exact.

In particular, by Lemma 5.5.5, the hypothesis holds if either p is surjective and E is a Kan complex, or if B is a Kan complex.

Sketch of Proof. Observe that p may be regarded as a simplicial map of triples $(E, F, \langle e\rangle) \to (B, \langle b\rangle, \langle b\rangle)$, thus inducing homomorphisms $\pi_n(p)^\#\colon \pi_n(E, F, \langle e\rangle) \to \pi_n(B, \langle b\rangle)$. Now let $[x] \in \pi_n(B, \langle b\rangle)$ and $x \in B_n$ be some representative of $[x]$, then consider the map $x\colon \Delta^n \to B$ representing x as per Lemma 2.1.4, and take the horn Λ^n_0 in E to be the constant map $c_{\langle e\rangle}$ at $\langle e\rangle$, to obtain the diagram

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{c_{\langle e\rangle}} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \xrightarrow[x]{} & B \end{array}$$

of solid arrows, which commutes as $d_i(x) = \langle b \rangle_{n-1}$ for each i, for it is a representative of $[x]$. Then as p is a fibration, there exists a dashed arrow which makes the resulting diagram commute. Namely, by commutativity, the dashed arrow $\Delta^n \to E$ represents an element in $\widehat{x} \in E_n$ with $d_0(\widehat{x}) \in F_{n-1}$, and for each i, $d_i(d_0(\widehat{x})) = \langle e \rangle_{n-2}$. In particular, $d_0(\widehat{x})$ is a representative for an element $[d_0(\widehat{x})] \in \pi_{n-1}(F, \langle e \rangle)$. Now with the same notations, define maps[47]

$$\begin{aligned} q_n \colon \pi_n(B, \langle b \rangle) &\to \pi_n(E, F, \langle e \rangle) \\ [x] &\mapsto [\widehat{x}] \end{aligned} \quad \text{and} \quad \begin{aligned} \partial_n^{\#} \colon \pi_n(B, \langle b \rangle) &\to \pi_{n-1}(F, \langle e \rangle) \\ [x] &\mapsto [d_0(\widehat{x})]. \end{aligned}$$

One may verify that these define homomorphisms of groups for each $n \geq 1$, and that q_n is the inverse of $\pi_n(p)^{\#}$. Finally the diagram

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\pi_{n+1}(i)} & \pi_{n+1}(E, \langle e \rangle) & \xrightarrow{\pi_{n+1}(j)} & \pi_{n+1}(E, F, \langle e \rangle) & \xrightarrow{\partial^{\pi}_{n+1}} & \pi_n(F, \langle e \rangle) & \xrightarrow{\pi_n(i)} & \pi_n(E, \langle e \rangle) & \xrightarrow{\pi_n(j)} & \cdots \\ & & \| & & \downarrow{\scriptstyle \pi_{n+1}(p)^{\#}} & & \| & & \| & & \\ \cdots & \xrightarrow[\pi_{n+1}(i)]{} & \pi_{n+1}(E, \langle e \rangle) & \xrightarrow[\pi_{n+1}(p)]{} & \pi_{n+1}(B, \langle b \rangle) & \xrightarrow[\partial^{\#}_{n+1}]{} & \pi_n(F, \langle e \rangle) & \xrightarrow[\pi_n(i)]{} & \pi_n(E, \langle e \rangle) & \xrightarrow[\pi_n(j)]{} & \cdots \end{array}$$

commutes, and Lemma 5.4.11 implies the result. □

5.6 Hurewicz Theorem

Recall that the n^{th} homology $H_n(X)$ (from Remark 1.2.5) of a simplicial set X is defined to be the quotient of abelian groups $Z_n(C_\star(X))/B_n(C_\star(X))$ of n-**cycles**

$$Z_n(C_\star(X)) := \ker(\partial_n^C) \quad \text{by } n\text{-}\textbf{boundaries} \quad B_n(C_\star(X)) := \operatorname{img}(\partial_{n+1}^C),$$

of its Moore complex $C_\star(X)$ with boundary maps $\partial_n^C := \sum_{i=0}^{n} (-1)^i d_i \colon X_n \to X_{n-1}$.

Also recall that the **reduced homology** $\widetilde{H}_n(X)$ is the homology of the augmentation X^+ of X (as in Remark 1.2.5). The unique map $d^0 \colon \emptyset \hookrightarrow [\![1]\!]$ induces a map $X^+(d^0) := d_0 \colon X_0 \to X_{-1}$ which then induces a surjective map $\epsilon := C_\star(d_0)$. It follows by this that there are isomorphisms $\widetilde{H}_n(X) \cong H_n(X)$ for all $n \geq 1$ and $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ (hence the word "reduced").

Let $(X, A, \langle x \rangle)$ be a pointed Kan pair and $[y] \in \pi_n(X, \langle x \rangle)$. Then for y a representative for the class $[y]$, $d_i(y) = \langle x \rangle_{n-1}$ for each i and so $y \in Z_n(C_\star(X))$. Let $\overline{y}$ denote the homology class generated by y as per Notation 5.4.1.

Lemma 5.6.1 (Lemma 13.3 in [19])**.** With notations as above, the assignment $[y] \mapsto \overline{y}$ defines a well defined homomorphism of groups

$$h^1 \colon \pi_n(X, \langle x \rangle) \to \widetilde{H}_n(X).$$

Proof. (Well Defined) Suppose $e, e' \in X_n$ via a homotopy σ, then $\partial_{n+1}^C(\sigma) = (-1)^n(e - e')$ by the definition of ∂_{n+1}^C, thus e and e' are homologous.

(Group Homomorphism) Suppose $[y], [z] \in \pi_n(X, \langle x \rangle)$ and y and z are representatives for $[y]$ and $[z]$, respectively. Then consider the horn

$$\Lambda_n^{n+1} \xrightarrow{(\langle x \rangle_n, \ldots, \langle x \rangle_n, y, -, z)} X$$

exhibiting a filler $\sigma \in X_{n+1}$ by the Kan condition. Then $\partial_{n+1}^C(\sigma) = (-1)^{n+1}(y - d_n(\sigma) + z)$ thus $d_n(\sigma)$ is homologous to $y + z$, whence $d_n(\sigma)$ is a generator for the of the product of $[x]$ and $[y]$ in $\pi_n(X, \langle x \rangle)$ by construction. □

[47] In particular, these assignments determine the maps.

Continuing on from the discussion before Lemma 5.6.1, set $C_\star(X,A) := C_\star(X)/C_\star(A)$ and the homology of the Kan pair (X,A) to be $H_n(X,A){:=}H_n(C_\star(X,A))$. Suppose z represents some element $[z]$ in $\pi_n(X,A,\langle x\rangle)$, then $d_i(z) \in A_{n-1}$ for each i. Thus $\partial_n^C(z)$ is trivial in the group $C_{n-1}(X,A)$. Thus z is an n-cycle in $Z_n(C_\star(X,A))$. As before, let $\overline{z}$ denote the homology class of z.

Lemma 5.6.2 (Lemma 13.3 in [19])**.** With notations as above, the assignment $[z] \mapsto \overline{z}$ defines a well defined homomorphism of groups

$$h^2\colon \pi_n(X,A,\langle x\rangle) \to \widetilde{H}_n(X,A).$$

Proof. Similar to the proof of Lemma 5.6.1. □

(Proposition 13.4 in [19]) The homomorphisms h^1 and h^2 from Lemmas 5.6.1 and 5.6.2, respectively, are called **Hurewicz homomorphisms**. They define natural transformations from the functor π_n to the functor $\widetilde{H}_n$. Define the homomorphism $\partial_{n+1}^H\colon H_{n+1}(X,A) \to \widetilde{H}_n(A)$ to be the map that carries a class $\overline{z}$ with some representative $z \in Z_{n+1}(C_\star(X,A))$ to the class $\overline{d_0(z)}$, then by construction, there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\pi_{n+1}(i)} & \pi_{n+1}(X,\langle x\rangle) & \xrightarrow{\pi_{n+1}(j)} & \pi_{n+1}(X,A,\langle x\rangle) & \xrightarrow{\partial^\pi_{n+1}} & \pi_n(A,\langle x\rangle) & \xrightarrow{\pi_n(i)} & \pi_n(X,\langle x\rangle) & \xrightarrow{\pi_n(j)} & \cdots \\
 & & \downarrow h^1 & & \downarrow h^2 & & \downarrow h^2 & & \downarrow h^1 & & \\
\cdots & \xrightarrow[H_{n+1}(i)]{} & H_{n+1}(X) & \xrightarrow[H_{n+1}(j)]{} & H_{n+1}(X,A) & \xrightarrow[\partial^H_{n+1}]{} & \widetilde{H}_n(A) & \xrightarrow[H_n(i)]{} & \widetilde{H}_n(X) & \xrightarrow[H_n(j)]{} & \cdots .
\end{array}$$

Definition 5.6.3 (n-connected; Definition 3.6 in [19])**.** A pointed Kan complex $(X,\langle x\rangle)$ is called **n-connected** for $n \geq 0$ if $\pi_0(X,\langle x\rangle) = \{[x]\}$ and for each $1 \leq k \leq n$, the group $\pi_k(X,\langle x\rangle)$ is trivial.

Theorem 5.6.4 (Hurewicz Theorem; Theorems 13.5 and 13.6 in [19])**.** Let $n \geq 1$ be fixed, if $(X,\langle x\rangle)$ is an $(n-1)$-connected pointed Kan complex, then there is an isomorphism of groups

$$H_n(X) \cong \pi_n(X,\langle x\rangle)/[\pi_n(X,\langle x\rangle),\pi_n(X,\langle x\rangle)]$$

and in particular, when $n \geq 2$ by Proposition 5.4.5, $H_n(X) \cong \pi_n(X,\langle x\rangle)$.

The resemblance between the homotopy theory of topological spaces and that of Kan complexes can be formalised. In fact, there is an equivalence of homotopy categories $Ho(\mathbf{Kan}) \simeq Ho(\mathbf{CW})$. This comparison is now investigated.

6 Model Categories

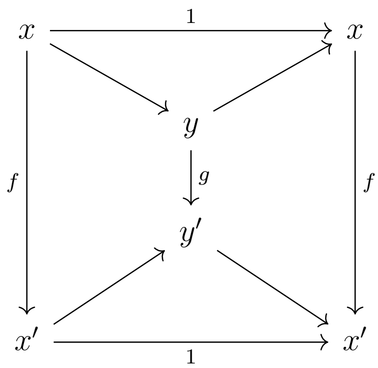
Loosely, a model category is complete category (see M1 in Definition 6.1.2) with additional structure determined by a choice of three subclasses of morphisms which are required to satisfy certain axioms. These axioms allow for "homotopy theory type" constructions on arbitrary complete categories satisfying them. In particular, for every model category $\mathbf{C}$, there is an associated homotopy category $Ho(\mathbf{C})$, and the notion of equivalence for model categories induces an equivalence of these homotopy categories.

This chapter will outline the prototypical examples of model categories on the categories $s\mathbf{Set}$ and **Top**. The aim of this chapter is to systematically summarise some of what has been covered for simplicial sets by formally comparing it with similar constructions for topological spaces.

Note. Some of the crucial details and technicalities have been omitted from this chapter. The reader is pointed to [5], [14] and [13] for some of the details. Explicit references will be provided as and when they are required.

6.1 Model Structures

Definition 6.1.1 (Retract; Definition 2.1 in [14])**.** Let f and g be morphisms in a category $\mathbf{C}$, f is called a **retract** of g if there exist unlabelled arrows such that the prism



commutes.

Definition 6.1.2 (Model Category; Definition 3.3 in [5])**.** A **model category** is a 4-tuple $(\mathbf{C}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ (sometimes shortened to $\mathbf{C}$) consisting of a category $\mathbf{C}$ with three subclasses of morphisms $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ of $\mathrm{mor}(\mathbf{C})$ called **fibrations**, **cofibrations** and **weak equivalences**, respectively, satisfying the following axioms.

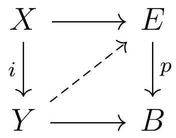
M0 The identity map on any object is contained in $\mathcal{F} \cap \mathcal{C} \cap \mathcal{W}$ and each of $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ are closed under compositions of morphisms within them.

M1 The category $\mathbf{C}$ is complete and cocomplete (i.e. limits and colimits over all diagrams $\mathbf{J} \to \mathbf{C}$ where $\mathbf{J}$ is a small category, in other words $\mathbf{C}$ has all small limits and colimits).

M2 The class of weak equivalences satisfies the **two out of three** property - for $f, g \in \mathrm{mor}(\mathbf{C})$ such that $g \circ f \in \mathrm{mor}(\mathbf{C})$, if any two of f, g or $g \circ f$ is a weak equivalence, then so is the third.

M3 Retracts preserve cofibrations, fibrations and weak equivalences, that is to say if f is a retract of g and g is a fibration, cofibration or a weak equivalence, then so is f.

M4 Given the diagram of solid arrows



in **C**, the depicted dashed arrow exists a dashed arrow making the resulting diagram commute whenever $(i, p) \in \mathcal{C} \times (\mathcal{F} \cap \mathcal{W})$ or $(i, p) \in (\mathcal{C} \cap \mathcal{W}) \times \mathcal{F}$.

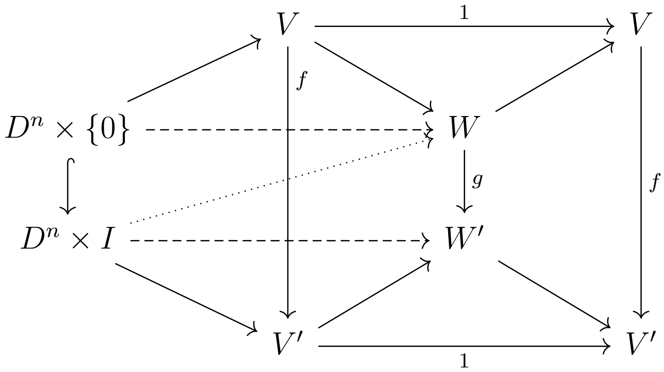
M5 Any morphism $f \in \mathrm{mor}(\mathbf{C})$ can be factored as $f = p_1 \circ i_1 = p_2 \circ i_2$ with $(i_1, p_1) \in \mathcal{C} \times (\mathcal{F} \cap \mathcal{W})$ and $(i_2, p_2) \in (\mathcal{C} \cap \mathcal{W}) \times \mathcal{F}$.

Sometimes maps in $\mathcal{F} \cap \mathcal{W}$ (respectively in $\mathcal{C} \cap \mathcal{W}$) are called **acyclic fibrations** (respectively **acyclic cofibrations**).

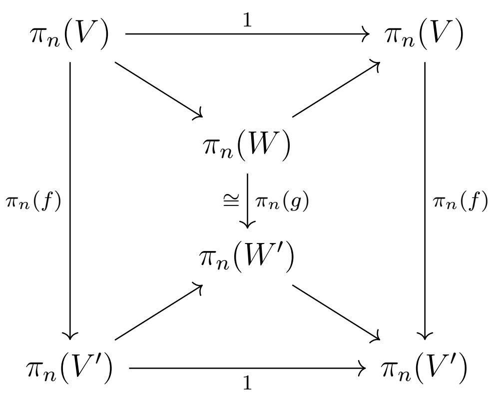
Example 6.1.3 ($\mathbf{Top}_Q$; Theorem 2.5 and §9 in [14])**.** The classical model category structure $\mathbf{Top}_Q := (\mathbf{Top}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ on the category **Top** of topological spaces is the model category with

- fibrations $\mathcal{F}$, the class of Serre fibrations (see Example 5.5.3).
- cofibrations $\mathcal{C}$, the class of retracts of inclusions $U \hookrightarrow V$, where V is obtained from U by attaching cells.
- weak equivalences $\mathcal{W}$, the class of weak homotopy equivalences.

M0 and M1 are clear and M2 is a straightforward check. For M3, suppose $f : V \to V'$ is a retract of $g : W \to W'$ and g is a Serre fibration and the solid diagram



commutes. Compose with the solid arrows from V and V' into W and W', respectively, to obtain the depicted dashed arrows. Then since g is a Serre fibration, there is a dotted arrow making the square (and indeed the whole diagram) commute. Finally composing the dotted arrow with the solid arrow from W to V implies that f is a Serre fibration, as the two horizontal arrows are both identity maps. Cofibrations are closed under retracts as retracts are closed under retracts. Now suppose instead that g is a weak equivalence, then the relevant prism commutes, and upon applying π_n, the diagram



commutes. But then it follows by commutativity of the top and bottom triangles that the unlabelled maps form pairs of mutually inverse isomorphisms on the top and on the bottom. Thus f is also a weak equivalence. For M4 and M5 see [14].

Example 6.1.4 ($s\mathbf{Set}_Q$; 11.1 in [5])**.** The classical model structure $s\mathbf{Set}_Q := (s\mathbf{Set}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ on the category $s\mathbf{Set}$ of simplicial sets is the model category with

- fibrations $\mathcal{F}$, the class of Kan fibrations.
- cofibrations $\mathcal{C}$, the class of (level-wise) injections in $\mathrm{mor}(s\mathbf{Set})$.
- weak equivalences $\mathcal{W}$, the class of maps $f\colon X \to Y$ such that $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence in $\mathbf{Top}$.

M0 is clear and M1 holds as $s\mathbf{Set}$, being a presheaf category, inherits small limits and colimits from $\mathbf{Set}$. M2 can be checked directly (the same argument as that for $\mathbf{Top}_Q$ would work). M3 for Kan fibrations and weak equivalences is similar to that of $\mathbf{Top}_Q$ and M3 for cofibration hols as retracts of injections are injections. The verifications for M4 and M5 are omitted. See Ch.1, §11, Theorem 11.3 in [9]. This was a result proven by Quillen in [21].

Example 6.1.5 ($s\mathbf{C}_Q$; 11.2 in [5])**.** More generally, if $\mathbf{C}$ is a concrete category - i.e. a category with a faithful functor $U\colon \mathbf{C} \to \mathbf{Set}$ (e.g. the forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$) and $\mathbf{C}$ has small limits and colimits, then there is a model category structure $s\mathbf{C}_Q := (s\mathbf{C}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ on the category $s\mathbf{C}$ of simplicial objects in $\mathbf{C}$, inherited from $s\mathbf{Set}_Q$. Let sU denote the functor that applies U level-wise, carrying the simplicial object $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ in the category $\mathbf{C}$ to the simplicial set $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathbf{C} \xrightarrow{U} \mathbf{Set}$. Then the model category $s\mathbf{C}_Q$ has

- fibrations $\mathcal{F}$, the class of morphisms $f \in \mathrm{mor}(\mathbf{C})$ such that $sU(f)$ is a Kan fibration.
- cofibrations $\mathcal{C}$, the class of morphisms $f \in \mathrm{mor}(\mathbf{C})$ such that for any given commutative diagram of solid arrows

  $$\begin{array}{ccc} X & \longrightarrow & E \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & B \end{array}$$

  where $p \in \mathcal{F} \cap \mathcal{W}$, there is a dashed arrow making the resulting diagram commute.[48]
- weak equivalences $\mathcal{W}$, the class of maps $f\colon X \to Y$ such that $|sU(f)|\colon |X| \to |Y|$ is a weak homotopy equivalence in $\mathbf{Top}$.

Remark 6.1.6 (11.2 in [5])**.** For the special case of the category $\mathbf{Ab}$ of abelian groups and the forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Set}$, the normalisation functor $N_\star\colon s\mathbf{Ab} \xrightarrow{\simeq} Ch_{\geq 0}(\mathbf{Ab})$ from the Dold-Kan correspondence in Chapter 3, being an equivalence, transfers the model category structure $s\mathbf{Ab}_Q$ on simplicial abelian groups to the category $Ch_{\geq 0}(\mathbf{Ab})$ of non-negatively graded chain complexes over abelian groups. Thus this provides a strong connection between homological algebra over abelian groups and the homotopy theory of simplicial abelian groups (see discussion in 11.2 in [5]).

Definition 6.1.7 ((Co)fibrant Objects; Remark 3.4 in [5])**.** Let $(\mathbf{C}, \mathcal{F}, \mathcal{C}, \mathcal{W})$ be a model category. Let $\mathrm{term}(\mathbf{C})$ and $\mathrm{init}(\mathbf{C})$ denote the terminal and initial objects in $\mathbf{C}$, respectively. An object A of $\mathbf{C}$ is called

[48] That is to say, f has the **left lifting property** with respect to acyclic fibrations.

(a) **fibrant** if the unique map $A \to \text{term}(\mathbf{C})$ to the terminal object of $\mathbf{C}$ is a fibration.

(b) **cofibrant** if the unique map $\text{init}(\mathbf{C}) \to A$ from the initial object of $\mathbf{C}$ is a cofibration.

where both $\text{init}(\mathbf{C})$ and $\text{term}(\mathbf{C})$ exist by M1. Two objects are called **weakly equivalent** if there is a weak equivalence between them.

Example 6.1.8 ($\mathbf{Top}_Q$; Example 3.5 in [5]). In the model category structure $\mathbf{Top}_Q$ on spaces from Example 6.1.3,

(a) all objects are fibrant, as for any space W, the map $W \to \{\star\}$ is a Serre fibration due to the existence of the retraction $D^n \times I \to D^n \times \{0\}$.

(b) cofibrant objects are retracts of generalised CW-complexes. Here the word "generalised" refers to omitting the requirement that cells are attached in increasing order of dimension.

Example 6.1.9 ($s\mathbf{Set}_Q$; Ch. I, §9, Introduction, pg. 42 in [9]). In the model category $s\mathbf{Set}_Q$ on simplicial sets from Example 6.1.4,

(a) fibrant objects are Kan complexes (see last line in Definition 5.5.1).[49]

(b) all objects are cofibrant as there is an inclusion from the empty simplicial set to any simplicial set.

6.2 Homotopy Category of a Model Category

The association of a model category to its homotopy category mentioned at the start of this chapter is now briefly described. It will be seen that the category $Ho(\mathbf{Kan})$ mentioned after Proposition 5.3.2 may be constructed using this association.

Lemma 6.2.1 (Definition 8.1.2 and Proposition 8.1.3 in [13]). Every object A in a model category $\mathbf{C}$ is weakly equivalent to a fibrant object $A^{\mathcal{F}}$, called a **fibrant replacement** of A, and a cofibrant object $A^{\mathcal{C}}$, called a **cofibrant replacement** of A.

Proof. By M5 the unique map $A \to \text{term}(\mathbf{C})$ to the terminal object can be factored as

$$A \xrightarrow{i} A^{\mathcal{F}} \xrightarrow{p} \text{term}(\mathbf{C})$$

where i is a weak equivalence and p is a fibration, and dually the unique map $\text{init}(\mathbf{C}) \to A$ can be factored as

$$\text{init}(\mathbf{C}) \xrightarrow{i'} A^{\mathcal{C}} \xrightarrow{p'} A$$

where i' is a cofibration and p' is a weak equivalence. □

It is a fact that a cofibrant replacement of a fibrant replacement $\left(A^{\mathcal{F}}\right)^{\mathcal{C}}$ of an object A in $\mathbf{C}$ is both fibrant and cofibrant and is weakly equivalent to A. This is not proved here however.

Definition 6.2.2 (Cylinder Object; Definition 4.2 and Definition in §4 on pg. 19 in [5]). For c an object in a model category $(\mathbf{C}, \mathcal{F}, \mathcal{C}, \mathcal{W})$, a **cylinder object** for c is an object $\text{cyl}(c)$ of $\mathbf{C}$, such that the codiagonal map $1_c \coprod 1_c \colon c \coprod c \to c$ factors as $c \coprod c \xrightarrow{i} \text{cyl}(c) \xrightarrow{p} c$ where p is a weak equivalence.

- A cylinder object $\text{cyl}(c)$ is called a **good** cylinder object if the map i in the factorisation is a cofibration.

[49] For this reason Kan complexes are also sometimes referred to as **fibrant simplicial sets**.

- A good cylinder object is called a **very good** cylinder object if the map p in the factorisation is a fibration.

where i and p are as per factorisations above. Two morphisms $f, g \in \mathbf{C}(c, d)$ are called **left homotopic** (written $f \overset{L}{\sim} g$) if there is a map $H\colon \mathrm{cyl}(c) \to d$ in $\mathrm{mor}(\mathbf{C})$ such that the diagram

$$\begin{array}{ccccc} c & & & & \\ \downarrow & & & \searrow^{f} & \\ c\coprod c & \xrightarrow{i} & \mathrm{cyl}(c) & \xrightarrow{H} & Y \\ \uparrow & & & \nearrow_{g} & \\ c & & & & \end{array}$$

where the vertical maps form the universal cocone defining $c \coprod c$, commutes.[50] In this situation H is called a **homotopy** between f and g. If in addition the cylinder $\mathrm{cyl}(c)$ is good (respectively very good), then the H is called a **good** (respectively **very good**) homotopy

Very good cylinder objects are guaranteed to exist by axiom MC5. However, neither of the types of cylinder objects are necessarily unique. Nonetheless, some standard choices are now shown.

Example 6.2.3 ($\mathbf{Top}_Q$; Example on pg. 19 in [5])**.** In the model category $\mathbf{Top}_Q$ on spaces, the **standard** cylinder object of a space W is the space $W \times I$ with the factorisation

$$W \coprod W \xrightarrow{i} W \times I \xrightarrow{p} W$$

where i is the inclusion at the ends of the interval and p is the projection onto the first factor.[51] $W \times I$ is a cylinder object as I is a contractible and hence p is a weak equivalence. However it is *not* good as W may not be a generalised CW-complex. Thus by the commutativity of the relevant diagram, $f \overset{L}{\sim} g$ via a homotopy H if and only if f is homotopic to g as a continuous function via the homotopy H.

Example 6.2.4 ($s\mathbf{Set}_Q$)**.** In the model category $s\mathbf{Set}_Q$ on simplicial sets, the **standard** cylinder object $\mathrm{cyl}(X)$ of a simplicial set X is the simplicial set $X \times \Delta^1$ with the factorisation

$$X \coprod X \xrightarrow{i} X \times \Delta^1 \xrightarrow{p} X$$

where i is the inclusion induced by the structure maps $1 \times N(d^i)\colon X \times \Delta^0 \to X \times \Delta^1$ for $i = 1, 2$, and p is the projection onto the first factor. It is a cylinder object as there are homeomorphisms $|X \times \Delta^1| \cong |X| \times |\Delta^1| \cong |X| \times I$ by Theorem 4.4.6 and Example 4.3.2 and as I is contractible.

- It is good as the map i in the factorisation is an inclusion.
- However, it is *not* very good as when X is the terminal simplicial set Δ^0, then by Example 2.2.6, Δ^1 is not a Kan complex, and hence the fibration p is not acyclic.

Thus Definition 5.3.1 is an example of a good left homotopy between simplicial maps.

This highlights a difference between the definition for homotopy of simplicial maps and that for continuous maps, from the point of view of model categories.

[50] There is also a dual notion of right homotopies in model categories, defined by factorisations of the diagonal map. Read further on this in 4.12 in [5].

[51] This is also where the term "cylinder object" comes from. The schematic picture of $W \times I$ resembles a cylinder.

Proposition 6.2.5 (Lemma 4.7 in [5]). For $\mathbf{C}$ a model category and $A, B \in \text{obj}(\mathbf{C})$, where A is cofibrant, the relation $\overset{L}{\sim}$ is an equivalence relation on $\mathbf{C}(A, B)$.

Proof. Omitted. See proof of Lemma 4.7 in [5]. □

Definition 6.2.6 (Homotopy Category; Definition 5.6 in [5]). If $\mathbf{C}$ is a model category, the **homotopy category** $Ho(\mathbf{C})$ of $\mathbf{C}$ is the category with

$$\text{obj}(Ho(\mathbf{C})) := \text{obj}(\mathbf{C}) \quad \text{and} \quad Ho(\mathbf{C})(A, B) := \mathbf{C}\left(\left(A^{\mathcal{F}}\right)^{\mathcal{C}}, \left(B^{\mathcal{F}}\right)^{\mathcal{C}}\right).$$ [52]

Remark 6.2.7 (Remark 5.7 in [5]). There is in fact a functor $\mathbf{C} \to Ho(\mathbf{C})$ which is the identity on objects where the map $\mathbf{C}(A, B) \to Ho(\mathbf{C})(A, B)$ is surjective when each of A and B are both fibrant and cofibrant. In this case, it also induces a bijection $Ho(\mathbf{C})(A, B) \cong \mathbf{C}(A, B)/\overset{L}{\sim}$. Indeed it suffices for this bijection that A is cofibrant and B is fibrant. Justification for this has been omitted (see 4.21 and 5.11 in [5]).

The homotopy category of $\mathbf{C}$ does not depend on the classes of fibrations and cofibration of $\mathbf{C}$. The functor in the above remark may as well be defined as the functor universal in inverting the class of weak equivalences, that is, the localisation at the class of weak equivalences (6.1 and 6.2 in [5]).

Example 6.2.8 ($Ho(\mathbf{Top}_Q)$). In the homotopy category $Ho(\mathbf{Top}_Q)$ on the model structure on $\mathbf{Top}$, the class of morphisms $Ho(\mathbf{Top}_Q)(V, W)$, where V is a retract of a (generalised) CW-complex, is naturally isomorphic to the homotopy classes of maps from V to W in the usual (topological) sense.

Example 6.2.9 ($Ho(s\mathbf{Set}_Q)$). In the model category $s\mathbf{Set}_Q$ on simplicial sets, the category $Ho(s\mathbf{Set}_Q)$ has simplicial sets as objects, and for a Kan complex Y, the set $Ho(s\mathbf{Set}_Q)(X, Y)$ is naturally isomorphic to (simplicial) homotopy classes of maps from X to Y. Note that the fact that this is well defined is due to Proposition 5.3.2, which may be seen as a special case of Proposition 6.2.5 when Y is fibrant.

Remark 6.2.10. Example 6.2.9 is related to the fact that simplicial homotopy groups were only defined for pointed Kan complexes whilst topological homotopy groups may be defined for all pointed spaces. The simplicial homotopy group of a general simplicial set X may be defined as the homotopy group of a Kan complex that is weakly equivalent to X. This might seem somewhat circular, however it is a valid generalisation of the definition that has been provided.

Using the definitions defined so far and the a few more supplementary lemmas, one may prove the following generalisation of Whitehead's theorem for model categories.

Theorem 6.2.11 (Whiteheads Theorem; Lemma 4.24 in [5]). In a model category $(\mathbf{C}, \mathcal{F}, \mathcal{C}, \mathcal{W})$, a weak equivalence $f\colon A \to B$ between objects A, B of $\mathbf{C}$ that are both fibrant and cofibrant is a homotopy equivalence in the sense that there is a morphism g such that $g \circ f \overset{L}{\sim} 1_A$ and $f \circ g \overset{L}{\sim} 1_B$.

Proof. The proof may be found as the proof of Lemma 4.24 in [5]. It requires some additional lemmas which haven't been provided here. □

[52] It is required to check that this definition is well defined, but this verification has been omitted here.

6.3 Quillen Equivalences

Quillen equivalence is the correct notion of an equivalence between model categories. Namely, if two model categories are Quillen equivalent, they possess the same information for the purpose of homotopy theory.

Definition 6.3.1 (Definitions 8.5.2 and 8.5.20 in [13])**.** If $(\mathbf{C}, \mathcal{F}_\mathbf{C}, \mathcal{C}_\mathbf{C}, \mathcal{W}_\mathbf{C})$ and $(\mathbf{D}, \mathcal{F}_\mathbf{D}, \mathcal{C}_\mathbf{D}, \mathcal{W}_\mathbf{D})$ are model categories an adjunction

$$F\colon \mathbf{C} \rightleftarrows \mathbf{D} : G$$

is called a **Quillen equivalence** if

- F preserves cofibrations and G preserves fibrations.[53]
- For all objects $C \in \text{obj}(\mathbf{C})$ and $D \in \text{obj}(\mathbf{D})$, a morphism $C \to G(D)$ is in the class $\mathcal{W}_\mathbf{C}$ if and only if the map $F(C) \to D$ corresponding to it via the adjunction $F \dashv G$, is in $\mathcal{W}_\mathbf{D}$.

An adjunction between model categories that satisfies the first condition is called a **Quillen adjunction**.

In particular, in a Quillen adjunction $F \dashv G$, the functors F and G preserve classes of cofibrant and fibrant objects, respectively. The following verifies the claim at the start of this section.

Theorem 6.3.2 (Theorem 9.7 in [5])**.** If

$$F\colon \mathbf{C} \rightleftarrows \mathbf{D} : G$$

is a Quillen adjunction between model categories $\mathbf{C}$ and $\mathbf{D}$, then there is an adjunction

$$\mathbb{L}F\colon Ho(\mathbf{C}) \rightleftarrows Ho(\mathbf{D}) : \mathbb{R}G$$

between their respective homotopy categories. If in addition the adjoint pair (F,G) is a Quillen equivalence, then $\mathbb{L}F$ and $\mathbb{R}G$ are Quasi-inverses to one another giving an equivalence of categories $Ho(\mathbf{C}) \simeq Ho(\mathbf{D})$.

Proof. The proof uses Ken Brown's Lemma and is provided after Lemma 9.9 in [5]. □

Remark 6.3.3 (Definition 9.1 in [5])**.** The functors $\mathbb{L}F$ and $\mathbb{R}G$ in Theorem 6.3.2 are called **left derived** and **right derived** functors of F and G, respectively. They are uniquely determined up to canonical natural equivalences by the functors F and G.

Example 6.3.4 (11.1 on pg. 52 in [5])**.** By Lemma 4.4.1, there is an adjunction

$$|\cdot|\colon s\mathbf{Set} \rightleftarrows \mathbf{Top} : Sing$$

which is a Quillen equivalence as

- by Example 5.5.3, the singular simplicial set functor $Sing$ preserves fibrations and by Theorem 4.4.4, it is evident that the geometric realisation $|\cdot|$ applied to an inclusion of simplicial sets, is a cellular map.
- for X a simplicial set and W a space, a map $f\colon X \to Sing(W)$ is a weak equivalence iff $|f|\colon |X| \to |Sing(W)|$ is a weak homotopy equivalence, which is iff the corresponding map $f^\perp\colon |X| \to W$ is a weak homotopy equivalence using the two out of three property, as the counit $|Sing(W)| \to W$ is a weak equivalence (see pg. 69 in [9] or on pg. 19 in [16]).

In particular, by Example 2.2.7 a singular complex is a Kan complex and Example 4.4.4 is the statement the geometric realisation of a simplicial set is a CW-complex.

By Theorem 6.3.2 and Example 6.3.4, the equivalence $Ho(\mathbf{Kan}) \simeq Ho(\mathbf{CW})$ is established.

[53] This condition is related to the fact that left adjoints preserve colimits and right adjoints preserve limits.

Conclusion

Key Results

In summary, some of the key conclusions at the end of each chapters were as follows

- In Chapter 1, an equivalent definition for a category enriched in simplicial sets was arrived at in two different ways.
- In Chapter 2, the image of all small categories in $s\mathbf{Set}$ under the nerve functor was characterised and depicted by a Venn diagram.
- In Chapter 3, the proof of the Dold-Kan correspondence (the equivalence $Ch_{\geq 0}(\mathbf{Ab}) \simeq s\mathbf{Ab}$) was outlined.
- In Chapter 4, Milnor's proof of the fact that under mild hypothesis, geometric realisation preserves categorical products ($|X \times Y| \cong |X| \times |Y|$), was highlighted and its useful consequences were stated.
- In Chapter 5, the homomorphisms in the Hurewicz theorem were constructed.
- In Chapter 6, the origin of the equivalence $Ho(\mathbf{CW}) \simeq Ho(\mathbf{Kan})$ was observed.

Other notable results encountered include, the three adjunction relations on $s\mathbf{Set}$ (observed in Definition 2.3.8, Lemma 4.4.1 and Equation 5.1), the long exact sequence of simplicial homotopy groups and whiteheads theorem for model categories.

Concluding Remarks and Further Readings

Hopefully by now, the reader has an idea about the wealth of connections and applications that simplicial sets have. However still, this report is no where near a full treatment towards the subject. It merely manages to scratch the surface of the theory of simplicial sets, where one may have developed interest in a certain topic covered. A lot of the content was chosen according to the preferences of the author. Arguably the most standard sources for learning about simplicial sets are [19] and [9]. They include far more than this report was able to cover. The approach in [19] is somewhat at a "point-set" level, whilst that in [9], being relatively new, follows categorical constructions. A great companion to read alongside these is [14]. The reader interested in reading further about model categories is highly recommended to read the exquisitely written text [5] and to refer to [13] for further details. As the nerve functor is fully faithful and the geometric realisation gives an equivalence of homotopy categories $Ho(\mathbf{Kan}) \simeq Ho(\mathbf{CW})$, various properties of certain algebraic objects (e.g. groups) are preserved under the composite $B = |\cdot| \circ N$. Applications of this are found in group cohomology. The reader interested in reading further on this is referred to [4]. The fundamental groupoid (from Example 2.3.6) is the composite of the adjoints of these functor going in the opposite directions $\mathbf{Top} \xrightarrow{Sing} s\mathbf{Set} \xrightarrow{h} \mathbf{Cat}$. This has various applications in topology. The reader interested in the exploring this further, is recommended to read Chapter 6 onwards in [3].

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