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# Homotopy and geometric perspectives on string topology

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In these lecture notes I will try to summarize some recent advances in the new area of study known as *string topology*. This subject was initiated in the beautiful paper of Chas and Sullivan [3], and has attracted the attention of many mathematicians over the last few years. In its most basic form, string topology is the study of differential and algebraic topological properties of paths and loops in a manifold.

Throughout this note  $M^n$  will denote a closed,  $n$ -dimensional, oriented manifold.  $LM$  will denote the free loop space,

$$LM = \text{Map}(S^1, M).$$

For  $D_1, D_2 \subset M$  closed submanifolds,  $\mathcal{P}_M(D_1, D_2)$  will denote the space of paths in  $M$  that start at  $D_1$  and end at  $D_2$ ,

$$\mathcal{P}_M(D_1, D_2) = \{\gamma : [0, 1] \rightarrow M, \gamma(0) \in D_1, \gamma(1) \in D_2\}.$$

The paths and loops we consider will always be assumed to be piecewise smooth. Such spaces of paths and loops are well known to be infinite dimensional manifolds, and roughly speaking, string topology is the study of the intersection theory in these manifolds.

Recall that for closed, oriented manifolds, there is an intersection pairing,

$$H_r(M) \times H_s(M) \rightarrow H_{r+s-n}(M)$$

which is defined to be Poincare dual to the cup product,

$$H^{n-r}(M) \times H^{n-s}(M) \xrightarrow{\cup} H^{2n-r-s}(M).$$

The geometric significance of this pairing is that if the homology classes are represented by submanifolds,  $P^r$  and  $Q^s$  with transverse intersection, then the image of the intersection pairing is represented by the geometric intersection,  $P \cap Q$ .

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The remarkable result of Chas and Sullivan says that even without Poincare duality, there is an intersection type product

$$\mu : H_p(LM) \times H_q(LM) \longrightarrow H_{p+q-n}(LM)$$

that is compatible with both the intersection product on  $H_*(M)$  via the map  $ev : LM \rightarrow M$  ( $\gamma \rightarrow \gamma(0)$ ), and with the Pontrjagin product in  $H_*(\Omega M)$ .

The construction of this pairing involves consideration of the diagram,

$$LM \xleftarrow{\gamma} Map(\mathbf{8}, M) \xrightarrow{e} LM \times LM. \quad (1)$$

Here  $Map(\mathbf{8}, M)$  is the mapping space from the figure 8 to  $M$ , which can be viewed as the subspace of  $LM \times LM$  consisting of those pairs of loops that agree at the basepoint.  $\gamma : Map(\mathbf{8}, M) \rightarrow LM$  is the map on mapping spaces induced by the pinch map  $S^1 \rightarrow S^1 \vee S^1$ .

Chas and Sullivan constructed this pairing by studying intersections of chains in loop spaces. A more homotopy theoretic viewpoint was taken by Cohen and Jones in [5] who viewed  $e : Map(\mathbf{8}, M) \rightarrow LM \times LM$  as an embedding, and showed there is a tubular neighborhood homeomorphic to a normal given by the pullback bundle,  $ev^*(TM)$ , where  $ev : LM \rightarrow M$  is the evaluation map mentioned above. They then constructed a Pontrjagin-Thom collapse map whose target is the Thom space of the normal bundle,  $\tau_e : LM \times LM \rightarrow Map(\mathbf{8}, M)^{ev^*(TM)}$ . Computing  $\tau_e$  in homology and applying the Thom isomorphism defines an ‘‘umkehr map’’,

$$e_! : H_*(LM \times LM) \rightarrow H_{*-n}(Map(\mathbf{8}, M)).$$

The Chas-Sullivan loop product is defined to be the composition

$$\mu_* = \gamma_* \circ e_! : H_*(LM \times LM) \rightarrow H_{*-n}(Map(\mathbf{8}, M)) \rightarrow H_{*-n}(LM).$$

Notice that the umkehr map  $e_!$  can be defined for a generalized homology theory  $h_*$  whenever one has a Thom isomorphism of the tangent bundle,  $TM$ , which is to say a generalized homology theory  $h_*$  for which the representing spectrum is a ring spectrum, and which supports an orientation of  $M$ .

By twisting the Pontrjagin-Thom construction by the virtual bundle  $-TM$ , one obtains a map of spectra,

$$\tau_e : LM^{-TM} \wedge LM^{-TM} \rightarrow Map(\mathbf{8}, M)^{ev^*(-TM)},$$

where  $LM^{-TM}$  is the Thom spectrum of the pullback of the virtual bundle  $ev^*(-TM)$ . Now we can compose, to obtain a multiplication

$$LM^{-TM} \wedge LM^{-TM} \xrightarrow{\tau_e} Map(\mathbf{8}, M)^{ev^*(-TM)} \xrightarrow{\gamma} LM^{-TM}.$$

The following was proved by Cohen and Jones in [5].

**Theorem 1.** *Let  $M$  be a closed manifold, then  $LM^{-TM}$  is a ring spectrum. If  $M$  is orientable the ring structure on  $LM^{-TM}$  induces the Chas-Sullivan loop product on  $H_*(LM)$  by applying homology and the Thom isomorphism.*

The ring structure on the spectrum  $LM^{-TM}$  was also observed by Dwyer and Miller using different methods.

In [4], Cohen and Godin generalized the loop product in the following way. Observe that the figure 8 is homotopy equivalent to the pair of pants surface  $P$ , which we think of as a genus 0 cobordism between two circles and one circle.

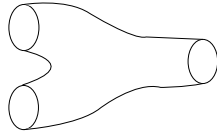


Figure 1: Pair of pants  $P$

Furthermore, diagram 1 is homotopic to the diagram of mapping spaces,

$$LM \xleftarrow{\rho_{out}} Map(P, M) \xrightarrow{\rho_{in}} (LM)^2$$

where  $\rho_{in}$  and  $\rho_{out}$  are restriction maps to the “incoming” and “outgoing” boundary components of the surface  $P$ . So the loop product can be viewed as a composition,

$$\begin{aligned} \mu = \mu_P &= (\rho_{out})_* \circ (\rho_{in})! : (H_*(LM))^{\otimes 2} \rightarrow H_*(Map(P, M)) \\ &\rightarrow H_*(LM) \end{aligned}$$

where using the figure 8 to replace the surface  $P$  can be viewed as a technical device that allows one to define the umkehr map  $(\rho_{in})!$ .

In general if one considers a surface of genus  $g$ , viewed as a cobordism from  $p$  incoming circles to  $q$  outgoing circles,  $\Sigma_{g,p+q}$ , one gets a similar diagram

$$(LM)^q \xleftarrow{\rho_{out}} Map(\Sigma_{g,p+q}, M) \xrightarrow{\rho_{in}} (LM)^p.$$

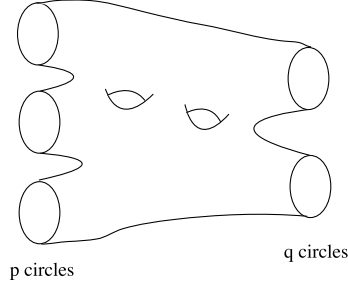


Figure 2:  $\Sigma_{g,p+q}$

In [4] Cohen and Godin used the theory of “fat” or “ribbon” graphs to represent surfaces as developed by Harer, Penner, and Strebel [11], [12], [14], in order to define Pontrjagin-Thom maps,

$$\tau_{\Sigma_{g,p+q}} : (LM)^p \rightarrow \text{Map}(\Sigma_{g,p+q}, M)^{\nu(\Sigma_{g,p+q})}$$

where  $\nu(\Sigma_{g,p+q})$  is the appropriately defined normal bundle of  $\rho_{in}$ . By applying (perhaps generalized) homology and the Thom isomorphism, they defined the umkehr map,

$$(\rho_{in})! : H_*((LM)^p) \rightarrow H_{*+\chi(\Sigma_{g,p+q})\cdot n}(\text{Map}(\Sigma_{g,p+q}, M)),$$

where  $\chi(\Sigma_{g,p+q}) = 2 - 2g - p - q$  is the Euler characteristic. Cohen and Godin then defined the string topology operation to be the composition,

$$\mu_{\Sigma_{g,p+q}} = \rho_{out} \circ (\rho_{in})! : H_*((LM)^p) \rightarrow H_{*+\chi(\Sigma_{g,p+q})\cdot n}(\text{Map}(\Sigma_{g,p+q}, M)) \rightarrow H_{*+\chi(\Sigma_{g,p+q})\cdot n}((LM)^q).$$

They proved that these operations respect gluing of surfaces,

$$\mu_{\Sigma_1 \# \Sigma_2} = \mu_{\Sigma_2} \circ \mu_{\Sigma_1}$$

where  $\Sigma_1 \# \Sigma_2$  is the glued surface as in figure 3 below.

The coherence of these operations are summarized in the following theorem.

**Theorem 2.** (Cohen-Godin [4]) *Let  $h_*$  be any multiplicative generalized homology theory that supports an orientation of  $M$ . Then the assignment*

$$\Sigma_{g,p+q} \longrightarrow \mu_{\Sigma_{g,p+q}} : h_*((LM)^p) \rightarrow h_*((LM)^q)$$

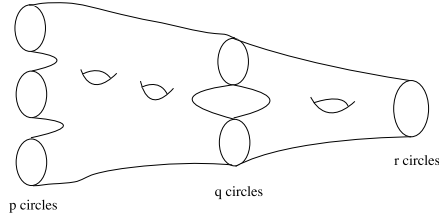


Figure 3:  $\Sigma_1 \# \Sigma_2$

is a positive boundary topological quantum field theory. “Positive boundary” refers to the fact that the number of outgoing boundary components,  $q$ , must be positive.

A theory with open strings was initiated by Sullivan [15] and developed further by A. Ramirez [13] and by Harrelson [10]. In this setting one has a collection of submanifolds,  $D_i \subset M$ , referred to as “D-branes”. This theory studies intersections in the path spaces  $P_M(D_i, D_j)$ .

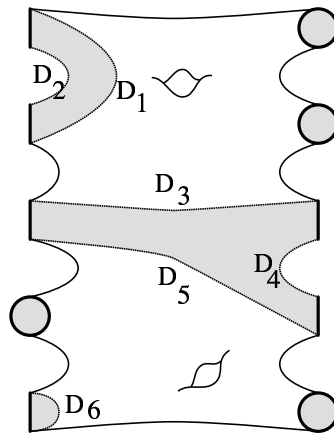


Figure 4: open-closed cobordism

A theory with  $D$ -branes involves “open-closed cobordisms” which are cobordisms between compact one dimensional manifolds whose boundary is partitioned into 3 parts:

1. Incoming circles and intervals

2. Outgoing circles and intervals

3. The rest is the “free boundary” which is itself a cobordism between the boundary of the incoming and boundary of the outgoing intervals. Each connected component of the “free boundary” is labelled by a  $D$ -brane. See figure 4.

In a topological field theory with  $D$ -branes, one associates to each boundary circle a vector space  $V_{S^1}$ , (in our case  $V_{S^1} = H_*(LM)$ ) and to an interval whose endpoints are labeled by  $D_i, D_j$ , one associates a vector space  $V_{D_i, D_j}$  (in our case  $V_{D_i, D_j} = H_*(P_M(D_i, D_j))$ ).

To an open-closed cobordism as above, one associates an operation from the tensor product of these vector spaces corresponding to the incoming boundaries to the tensor product of the vector spaces corresponding to the outgoing boundaries. Of course these operations have to respect the relevant gluing of open-closed cobordisms.

By developing a theory of fat graphs that encode the open-closed boundary data, Ramirez was able to prove that there are string topology operations that form a positive boundary, topological quantum field theory with  $D$ -branes. [13]

We end these notes by a discussion of three applications of string topology to classifying spaces of groups.

**Example 1:** *Application to Poincare duality groups.* This is work of H. Abbaspour, R.Cohen, and K. Gruher [2].

For  $G$  any discrete group, one has that the loop space of the classifying space satisfies

$$LBG \simeq \coprod_{[g]} BC_g$$

where  $[g]$  is the conjugacy class determined by  $g \in G$ , and  $C_g < G$  is the centralizer of  $g$ .

When  $BG$  is represented by a closed manifold, or more generally, when  $G$  is a Poincare duality group, the Chas-Sullivan loop product then defines pairings among the homologies of the centralizer subgroups. In [2] the authors describe this loop product entirely in terms of group homology, thus giving structure to the homology of Poincare-duality groups that had not been previously known.

**Example 2 :** *Applications to 3-manifolds.* This is work of H. Abbaspour [1]

Let  $\iota : H_*M \rightarrow H_*(LM)$  be induced by inclusion of constant loops. This is a split injection of rings. Write  $H_*(LM) = H_*(M) \oplus A_M$ . We say  $H_*(LM)$  has nontrivial extended loop products if the composition

$$A_M \otimes A_M \hookrightarrow H_*(LM) \otimes H_*(LM) \xrightarrow{\mu} H_*(LM)$$

is nontrivial.

Let  $M$  be a closed, irreducible 3-manifold. In a remarkable piece of work, Abbaspour showed the relationship between having a trivial extended loop product and  $M$  being “algebraically hyperbolic”.

This means that  $M$  is a  $K(\pi, 1)$  and its fundamental group has no rank 2 abelian subgroup. (If geometrization conjecture is true, this is equivalent to  $M$  admitting a complete hyperbolic metric.)

**Example 3:** *The string topology of classifying spaces of compact Lie groups.* This is work of K. Gruher [8] and of Gruher and Salvatore [9].

The goal of Gruher's work is to construct string topological invariants of  $LBG \simeq EG \times_G G$ , where  $G$  acts on itself via conjugation. Ultimately, one would like to understand the relationship between this structure and the work of Freed, Hopkins, Teleman [6] on twisted equivariant  $K$ -theory,  $K_G^T(G)$  and the Verlinde algebra.

The first observation in this program was to notice that the key ingredient in the forming of the Chas-Sullivan loop product is that the fibration  $ev : LM \rightarrow M$  is a fiberwise monoid over a closed oriented manifold. The fiber is  $\Omega M$ , which has the usual Pontrjagin product.

The following was proved by Gruher and Salvatore:

**Lemma 3.** *Let  $G \rightarrow E \rightarrow M$  be a fiberwise monoid over a closed manifold  $M$ . Then  $E^{-TM}$  is a ring spectrum.*

The following construction gives a large supply of examples of such fiberwise monoids over manifolds.

Let  $G \rightarrow P \rightarrow M$  be a principal  $G$  bundle over a closed manifold  $M$ . We can construct the corresponding adjoint bundle,

$$Ad(P) = P \times_G G \rightarrow M.$$

It is an easy observation that  $G \rightarrow Ad(P) \rightarrow M$  is a fiberwise monoid.

**Theorem 4.**  *$Ad(P)^{-TM}$  is a ring spectrum. This ring structure is natural with respect to maps of principal  $G$ -bundles.*

Let  $BG$  be classifying space of compact Lie groups. It is possible to construct a filtration of  $BG$ ,

$$M_1 \hookrightarrow M_2 \hookrightarrow \dots \hookrightarrow M_i \subset M_{i+1} \hookrightarrow \dots \hookrightarrow BG$$

where the  $M_i$ 's are compact, closed manifolds. An example of this is filtering  $BU(n)$  by Grassmannians.

Let  $G \rightarrow P_i \rightarrow M_i$  be the restriction of  $EG \rightarrow BG$ . By the above theorem one obtains an inverse system of ring spectra

$$P_1^{-TM_1} \leftarrow P_2^{-TM_2} \leftarrow \dots \leftarrow P_i^{-TM_i} \leftarrow P_{i+1}^{-TM_{i+1}} \leftarrow \dots$$

**Theorem 5.** *The homotopy type of this pro-ring-spectrum is a well defined invariant of  $BG$ . It is referred to as the "string topology of  $BG$ ".*

*Work in progress by Gruher:* Apply  $K_*$  to this pro-ring spectrum, as well as twisted  $K$ -theory. Relate the string topology multiplicative structure to the multiplicative structure on  $K_G^\tau(G)$  found by Freed, Hopkins, and Teleman, known as the “fusion product” in the Verlinde algebra.

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# NEW FORMS OF $K$ -THEORY

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The main objective of this paper is to introduce a new geometrically defined cohomology theory that essentially is of chromatic filtration 2. It may be viewed as a geometrically defined form of elliptic cohomology or as a second order  $K$ -theory. See [BDR04] for a more comprehensive treatment.

The theory that is presented is represented by the algebraic  $K$ -theory spectrum  $K(\mathcal{V})$  of the Kapranov–Voevodsky 2-category of 2-vector spaces [KV94]. A 2-vector space is much like a complex vector space, but with all occurrences of complex numbers, sums, products and equalities replaced by finite-dimensional complex vector spaces, direct sums, tensor products and coherent isomorphisms, respectively. It is geometrically defined in the sense that the 0-th cohomology group  $K(\mathcal{V})^0(X)$  of a space  $X$  can be defined in terms of equivalence classes of *2-vector bundles* over  $X$  (or more precisely, over the total space  $Y$  of a Serre fibration  $Y \rightarrow X$  with acyclic homotopy fibers, i.e., an acyclic fibration). A 2-vector bundle over  $X$  is a suitable bundle of categories, defined much like a complex vector bundle over  $X$ , but subject to the same replacements as above. The previously studied notion of a gerbe over  $X$  with band  $\mathbb{C}^*$  is a special case of a 2-vector bundle, corresponding in the same way to a complex line bundle.

The spectrum  $K(\mathcal{V})$  is equivalent to the algebraic  $K$ -theory spectrum  $K(ku)$  of the connective topological  $K$ -theory spectrum  $ku$ , considered as a “brave new ring”, i.e., as an  $\mathbb{S}$ -algebra. This is a special case of a more general conjecture, where for a symmetric bimonoidal category  $\mathcal{B}$  (which is a generalization of a commutative semi-ring) we compare the category of finitely generated free modules over  $\mathcal{B}$  to the category of finitely generated free modules over the commutative  $\mathbb{S}$ -algebra  $\mathbb{A} = \text{Spt}(\mathcal{B})$  (which is a generalization of a commutative ring) associated to  $\mathcal{B}$ . This amounts to a form of “positive thinking”, asserting that for the purpose of forming algebraic  $K$ -theory spectra it should not matter whether we start with a semi-ring-like object (such as the symmetric bimonoidal category  $\mathcal{B}$ ) or the ring-like object given by its additive Grothendieck group completion (such as the commutative  $\mathbb{S}$ -algebra  $\mathbb{A}$ ).

We know that  $K(ku)$ , or rather a spectrum very closely related to it, is essentially of chromatic filtration 2. In this sense we shall allow ourselves to think of  $K(ku)$ , and  $K(\mathcal{V})$ , as a connective form of elliptic cohomology.

Basically we define a new form of  $K$ -theory (2- $K$  or second order  $K$ -theory) based on the new notion of a 2-vector bundle. We find it useful to consider two types: chartered and represented 2-vector bundles. The definitions are as follows:

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The present paper is a summary of the talk I gave at the Conference of Pure and Applied Topology at the Isle of Skye on June 21st 2005.

**Definition 1.** Let  $X$  be a topological space. An *ordered open cover*  $(\mathcal{U}, \mathcal{J})$  of  $X$  is a collection  $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{J}\}$  of open subsets  $U_\alpha \subset X$ , indexed by a partially ordered set  $\mathcal{J}$ , such that

- (1) the  $U_\alpha$  cover  $X$  in the sense that  $\bigcup_\alpha U_\alpha = X$ , and
- (2) the partial ordering on  $\mathcal{J}$  restricts to a total ordering on each finite subset  $\{\alpha_0, \dots, \alpha_p\}$  of  $\mathcal{J}$  for which the intersection  $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$  is nonempty.

The partial ordering on  $\mathcal{J}$  makes the nerve of the open cover  $\mathcal{U}$  an ordered simplicial complex, rather than just a simplicial complex. We say that  $\mathcal{U}$  is a *good cover* if each finite intersection  $U_{\alpha_0 \dots \alpha_p}$  is either empty or contractible.

**Definition 2.** Let  $X$  be a topological space, with an ordered open cover  $(\mathcal{U}, \mathcal{J})$ , and let  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$  be a non-negative integer. A *charted 2-vector bundle*  $\mathcal{E}$  of rank  $n$  over  $X$  consists of

- (1) an  $n \times n$  matrix

$$E^{\alpha\beta} = (E_{ij}^{\alpha\beta})_{i,j=1}^n$$

of complex vector bundles over  $U_{\alpha\beta}$ , for each pair  $\alpha < \beta$  in  $\mathcal{J}$ , such that over each point  $x \in U_{\alpha\beta}$  the integer matrix of fiber dimensions

$$\dim(E_x^{\alpha\beta}) = (\dim E_{ij,x}^{\alpha\beta})_{i,j=1}^n$$

is invertible, i.e., has determinant  $\pm 1$ , and

- (2) an  $n \times n$  matrix

$$\phi^{\alpha\beta\gamma} = (\phi_{ik}^{\alpha\beta\gamma})_{i,k=1}^n : E^{\alpha\beta} \cdot E^{\beta\gamma} \xrightarrow{\cong} E^{\alpha\gamma}$$

of vector bundle isomorphisms

$$\phi_{ik}^{\alpha\beta\gamma} : \bigoplus_{j=1}^n E_{ij}^{\alpha\beta} \otimes E_{jk}^{\beta\gamma} \xrightarrow{\cong} E_{ik}^{\alpha\gamma}$$

over  $U_{\alpha\beta\gamma}$ , for each triple  $\alpha < \beta < \gamma$  in  $\mathcal{J}$ , such that

- (3) the diagram

$$\begin{array}{ccc} E^{\alpha\beta} \cdot (E^{\beta\gamma} \cdot E^{\gamma\delta}) & \xrightarrow{\underline{\alpha}} & (E^{\alpha\beta} \cdot E^{\beta\gamma}) \cdot E^{\gamma\delta} \\ \text{id} \cdot \phi^{\beta\gamma\delta} \downarrow & & \downarrow \phi^{\alpha\beta\gamma} \cdot \text{id} \\ E^{\alpha\beta} \cdot E^{\beta\delta} & \xrightarrow{\phi^{\alpha\beta\delta}} & E^{\alpha\delta} \xleftarrow{\phi^{\alpha\gamma\delta}} E^{\alpha\gamma} \cdot E^{\gamma\delta} \end{array}$$

of vector bundle isomorphisms over  $U_{\alpha\beta\gamma\delta}$  commutes, for each chain  $\alpha < \beta < \gamma < \delta$  in  $\mathcal{J}$ .

Here  $\underline{\alpha}$  denotes the (coherent) natural associativity isomorphism for the matrix product  $\cdot$  derived from the tensor product  $\otimes$  of vector bundles. We call the  $n \times n$  matrices  $E^{\alpha\beta}$  and  $\phi^{\alpha\beta\gamma}$  the *gluing bundles* and the *coherence isomorphisms* of the charted 2-vector bundle  $\mathcal{E} \downarrow X$ , respectively. We also define a notion of equivalence of two-vector bundles.

For a symmetric bimonoidal category  $\mathcal{B}$  we introduce the monoidal category  $M_n(\mathcal{B})$  of matrices over  $\mathcal{B}$  and suitably  $GL_n(\mathcal{B})$ , the monoidal category of weakly invertible matrices or weak equivalences  $\mathcal{B}^n \simeq \mathcal{B}^n$ .

**Definition 3.** Let  $\mathcal{B}$  be a symmetric bimonoidal category. The *algebraic  $K$ -theory* of the 2-category of (finitely generated free) modules over  $\mathcal{B}$  is the loop space

$$K(\mathcal{B}) = \Omega B\left(\coprod_{n \geq 0} |BGL_n(\mathcal{B})|\right).$$

Here  $|BGL_n(\mathcal{B})|$  is the geometric realization of the bar construction on the monoidal category  $GL_n(\mathcal{B})$  of weakly invertible  $n \times n$  matrices over  $\mathcal{B}$ . (See [BDR04].) The block sum of matrices  $GL_n(\mathcal{B}) \times GL_m(\mathcal{B}) \rightarrow GL_{n+m}(\mathcal{B})$  makes the coproduct  $\coprod_{n \geq 0} |BGL_n(\mathcal{B})|$  a topological monoid. The looped bar construction  $\Omega B$  provides a group completion of this topological monoid.

When  $\mathcal{B} = \mathcal{V}$  is the category of finite dimensional complex vector spaces, the (finitely generated free) modules over  $\mathcal{V}$  are called *2-vector spaces*, and  $K(\mathcal{V})$  is the algebraic  $K$ -theory of the 2-category of 2-vector spaces.

Let  $X$  be a topological space, with an ordered open cover  $(\mathcal{U}, \mathcal{J})$ . Recall that all morphisms in  $\mathcal{V}$  are isomorphisms, so  $\text{Ar } GL_n(\mathcal{V}) = \text{Iso } GL_n(\mathcal{V})$ .

**Definition 4.** A *represented 2-vector bundle*  $\mathcal{E}$  of rank  $n$  over  $X$  consists of

- (1) a *gluing map*

$$g^{\alpha\beta} : U_{\alpha\beta} \rightarrow |GL_n(\mathcal{V})|$$

for each pair  $\alpha < \beta$  in  $\mathcal{J}$ , and

- (2) a *coherence map*

$$h^{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow |\text{Ar } GL_n(\mathcal{V})|$$

satisfying  $s \circ h^{\alpha\beta\gamma} = g^{\alpha\beta} \cdot g^{\beta\gamma}$  and  $t \circ h^{\alpha\beta\gamma} = g^{\alpha\gamma}$  over  $U_{\alpha\beta\gamma}$ , for each triple  $\alpha < \beta < \gamma$  in  $\mathcal{J}$ , such that

- (3) the *2-cocycle condition*

$$h^{\alpha\gamma\delta} \circ (h^{\alpha\beta\gamma} \cdot \text{id}) \circ \underline{\alpha} = h^{\alpha\beta\delta} \circ (\text{id} \cdot h^{\beta\gamma\delta})$$

holds over  $U_{\alpha\beta\gamma\delta}$  for all  $\alpha < \beta < \gamma < \delta$  in  $\mathcal{J}$ .

There is a suitably defined notion of *equivalence* of represented 2-vector bundles, which we omit to formulate here.

**Definition 5.** Let  $2\text{-Vect}_n(X)$  be the set of equivalence classes of 2-vector bundles of rank  $n$  over  $X$ . For path-connected  $X$  let  $2\text{-Vect}(X) = \coprod_{n \geq 0} 2\text{-Vect}_n(X)$ . Whitney sum defines a pairing that makes  $2\text{-Vect}(X)$  an abelian monoid.

**Theorem 6.** *Let  $X$  be a finite CW complex. There are natural bijections*

$$2\text{-Vect}_n(X) \cong [X, |BGL_n(\mathcal{V})|]$$

and

$$2\text{-Vect}(X) \cong [X, \coprod_{n \geq 0} |BGL_n(\mathcal{V})|].$$

A *virtual 2-vector bundle* over  $X$  is described by an acyclic fibration  $a: Y \rightarrow X$  and a 2-vector bundle  $\mathcal{E} \downarrow Y$ . We write  $\mathcal{E} \downarrow Y \xrightarrow{a} X$ .

The following result says that formal differences of virtual 2-vector bundles over  $X$  are the geometric objects that constitute cycles for the contravariant homotopy functor represented by the algebraic  $K$ -theory space  $K(\mathcal{V})$ .

**Theorem 7.** *Let  $X$  be a finite CW complex. There is a natural group isomorphism*

$$\operatorname{colim}_{a: Y \rightarrow X} \operatorname{Gr}(2\text{-Vect}(Y)) \cong [X, K(\mathcal{V})]$$

where  $a: Y \rightarrow X$  ranges over the category of acyclic fibrations over  $X$ . Restricted to  $\operatorname{Gr}(2\text{-Vect}(X))$  (with  $a = \operatorname{id}$ ) the isomorphism extends the canonical monoid homomorphism  $2\text{-Vect}(X) \cong [X, \coprod_{n \geq 0} |BGL_n(\mathcal{V})|] \rightarrow [X, K(\mathcal{V})]$ .

Is the contravariant homotopy functor  $X \mapsto [X, K(\mathcal{V})] = K(\mathcal{V})^0(X)$  part of a cohomology theory, and if so, what is the spectrum representing that theory?

The topological symmetric bimonoidal category  $\mathcal{V}$  plays the role of a generalized commutative semi-ring in our definition of  $K(\mathcal{V})$ .

The passage from modules over the semi-ring object  $\mathcal{V}$  to modules over the ring object  $ku$  corresponds to maps  $|GL_n(\mathcal{V})| \rightarrow \widehat{GL}_n(ku)$  (stable equivalences  $\mathbb{A}^n \rightarrow \mathbb{A}^n$ ) and a map  $K(\mathcal{V}) \rightarrow K(ku)$ .

**Theorem 8.** *There is a weak equivalence  $K(\mathcal{V}) \simeq K(ku)$ . More generally,  $K(\mathcal{B}) \simeq K(\mathbb{A})$  for each symmetric bimonoidal category  $\mathcal{B}$  with associated commutative  $\mathbb{S}$ -algebra  $\mathbb{A} = \operatorname{Spt}(\mathcal{B})$*

A proof of this is now in the process of being written up.

The theorem asserts that the contravariant homotopy functor  $X \mapsto [X, K(\mathcal{V})]$  with 0-cycles given by the virtual 2-vector bundles over  $X$  is the 0-th cohomology group for the cohomology theory represented by the spectrum  $K(ku)$  given by the algebraic  $K$ -theory of connective topological  $K$ -theory. We consider the virtual 2-vector bundles over  $X$  to be sufficiently geometric objects (like complex vector bundles), that this cohomology theory then admits as geometric an interpretation as the classical examples of de Rham cohomology, topological  $K$ -theory and complex bordism.

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# CALCULUS OF FUNCTORS, OPERAD FORMALITY, AND EMBEDDING SPACES

GREGORY ARONE

ABSTRACT. Let  $M$  be a manifold and  $V$  a Euclidean space. We use calculus of functors to study the rational homology and homotopy of the space  $\overline{\text{Emb}}(M, V)$ , which is defined to be the homotopy fiber of the map  $\text{Emb}(M, V) \rightarrow \text{Imm}(M, V)$ . If  $M$  is one-dimensional, we get, essentially, the space of knots in  $\mathbb{R}^n$ . One general point that we want to make is that quite a bit of what had been done for knots can be done for general embedding spaces. Technically, our main theme is the interplay between embedding calculus, orthogonal calculus, and the rational formality of the little balls operad. The formality theorem implies that the spectral sequence for  $H_*(\overline{\text{Emb}}(M, V); \mathbb{Q})$  that arises from embedding calculus collapses at  $E_2$ . We get an even better statement when we translate this into the framework of orthogonal calculus. Namely, the orthogonal calculus Taylor tower of  $\mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V)$  splits as the product of its layers.

We write explicit formulas for the layers in the orthogonal towers of the functors  $\mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V)$  and  $\overline{\text{Emb}}(M, V)$  as a twisted cohomology of certain spaces of trees and graphs “grafted” on to  $M$ . The formulas show, in particular, that the homotopy groups of the layers depend only on the homology of  $M$ . Combining this with the splitting of the orthogonal tower, we conclude that as long as  $2\dim(M) + 1 < \dim(V)$ , the rational homology groups of  $\overline{\text{Emb}}(M, V)$  depend only on the rational homology of  $M$ .

Finally, we show that there also are collapsing results for spectral sequences computing the rational homotopy of  $\overline{\text{Emb}}(M, V)$ . For example, the rational homotopy spectral sequence arising from orthogonal calculus collapses at  $E_1$ . It follows that if  $\dim(M) + 2 < \dim(V)$  then the rational homotopy groups of  $\overline{\text{Emb}}(M, V)$  (in positive dimensions) depend only on the rational homology of  $M$ . The main ingredient of the proof here is the coformality of the little balls operad.

This is a report on joint work with Pascal Lambrechts and Ismar Volić.

Let  $M$  be a smooth manifold (possibly with a boundary) of dimension  $m$  and let  $V$  denote a Euclidean space, so  $V \cong \mathbb{R}^N$  for some  $N$ . We are interested in the space of embeddings  $\text{Emb}(M, V)$  or rather the space  $\overline{\text{Emb}}(M, V) := \text{hofiber}(\text{Emb}(M, V) \rightarrow \text{Imm}(M, V))$ , where  $\text{Imm}(M, V)$  denotes the space of immersions of  $M$  into  $V$ . Most of the time, we will work with the suspension spectrum  $\Sigma^\infty \overline{\text{Emb}}(M, V)$ , and our results are actually about the rationalization of this spectrum. In other words, our results are about the rational homology (and sometimes rational homotopy) of  $\overline{\text{Emb}}(M, V)$ .

Our main tools are calculus of functors and the formality of the little balls operad. There are two versions of calculus of functors (both developed by M. Weiss) that are relevant to us: embedding calculus and orthogonal calculus. Embedding calculus [4, 1] is the more elementary of the two, as it relies on familiar constructions from category theory. Embedding calculus is designed for studying contravariant functors on manifolds, such as  $F(M) = \text{Emb}(M, V)$ . Let  $\mathcal{O}_\infty(M)$  be the category of open subsets of  $M$  that are homeomorphic to a finite union of open balls. Morphisms are inclusions, and  $F$  is assumed to be a contravariant functor from  $\mathcal{O}_\infty(M)$  to spaces. The category  $\mathcal{O}_\infty(M)$  is filtered by an increasing sequence of full subcategories  $\mathcal{O}_k(M)$ ,

$k = 1, 2, \dots$ , where the objects of  $\mathcal{O}_k(M)$  are subsets of  $M$  homeomorphic to a disjoint union of at most  $k$  open balls in  $M$ . This sequence of subcategories gives rise to a tower of fibrations

$$(1) \quad F(M) \rightarrow T_\infty F(M) \longrightarrow \cdots \longrightarrow T_k F(M) \longrightarrow T_{k-1} F(M) \longrightarrow \cdots \longrightarrow T_1 F(M)$$

where

$$T_k F(M) := \operatorname{holim}_{U \in \mathcal{O}_k(M)} F(U).$$

In categorical terms,  $T_k F$  is the homotopy right Kan extension of the restriction of  $F$  to  $\mathcal{O}_k(M)$ . In calculus parlance,  $T_k F$  is the  $k$ -th Taylor polynomial of  $F$ . Indeed,  $T_k F$  is polynomial of degree  $k$  in a certain reasonable sense, and the natural transformation  $F \rightarrow T_k F$  is initial among natural transformations from  $F$  to polynomial functors of degree  $k$ .

The functor that interests us most is  $F(U) = \mathbf{H}\mathbb{Q} \wedge \overline{\operatorname{Emb}}(U, V)$ . If  $U$  is homeomorphic to a disjoint union of finitely many open balls, say  $U \cong k_U \times D^m$ , then  $\overline{\operatorname{Emb}}(U, V)$  is homotopy equivalent to the configuration space  $C(k_U, V)$  of  $k_U$ -tuples of distinct points in  $V$  (or, equivalently, the space of  $k_U$ -tuples of disjoint balls in  $V$ ). Abusing notation, we can write that

$$(2) \quad T_k \mathbf{H}\mathbb{Q} \wedge \overline{\operatorname{Emb}}(M, V) := \operatorname{holim}_{U \in \mathcal{O}_k(M)} \mathbf{H}\mathbb{Q} \wedge \overline{\operatorname{Emb}}(U, V) \simeq \operatorname{holim}_{U \in \mathcal{O}_k(M)} \mathbf{H}\mathbb{Q} \wedge C(k_U, V)$$

The right hand side in the above formula is not well-defined as written, but it gives the right idea. The formula tells us that under favorable circumstances<sup>1</sup>, the spectrum  $\mathbf{H}\mathbb{Q} \wedge \overline{\operatorname{Emb}}(M, V)$  can be written as a homotopy inverse limit of spectra of the form  $\mathbf{H}\mathbb{Q} \wedge C(k, V)$ . It is quite clear that the maps in the diagram are closely related to the structure map in the little balls' operad. Therefore, information about the stable rational homotopy type of the little balls operad may yield information about the rational homology of spaces of embeddings. We say that an operad  $\{E_n\}$  of spaces is formal if there is a quasi-isomorphisms of operads  $\Omega_*(E_n) \cong H_*(E_n)$ . Let  $B(n, V)$  be the space of disjoint  $n$ -tuples of balls in  $V$ . The spaces  $\{B(n, V)\}_{n=1}^\infty$  form the little balls operad. A key theorem of Kontsevich ([2], and see also [3]) asserts that this operad is formal

**Theorem 0.1.** *The operad  $\{B(n, V)\}$  is formal.*

From this, we can deduce our first theorem.

**Theorem 0.2.** *The diagram in the middle part of (2), which we used to define  $T_k \mathbf{H}\mathbb{Q} \wedge \overline{\operatorname{Emb}}(M, V)$ , is formal.*

The following theorem is an easy consequence.

**Theorem 0.3.** *The spectral sequence for calculating the rational homology of  $\overline{\operatorname{Emb}}(M, V)$  arising from embedding calculus collapses at  $E_2$ .*

Now we turn to the other brand of calculus that is relevant to us, namely Weiss' "orthogonal calculus" [5]. This is a calculus of covariant functors from the category of vector spaces and linear isometric inclusions to topological spaces (or spectra). With such a functor  $G$ , orthogonal calculus associates a tower of fibrations of functors  $P_n G(V)$ , where  $P_n G$  is the  $n$ -th Taylor polynomial of  $G$  in the orthogonal sense. As is customary, we let  $D_n G(V)$  denote the  $n$ -th homogeneous layer

<sup>1</sup>Specifically, if whenever the Taylor tower of the functor  $\mathbf{H}\mathbb{Q}(\overline{\operatorname{Emb}}(M, V))$  converges, which is ensured by the condition  $2 \dim(M) + 1 < \dim(V)$

in the orthogonal Taylor tower, namely the fiber of the map  $P_n G(V) \rightarrow P_{n-1} G(V)$ . The general theory says that  $D_n G(V)$  is determined by a spectrum with an action of  $O(n)$ , which we denote  $C_n$  and call the  $n$ -th derivative of  $G$ , and

$$(3) \quad D_n G(V) \simeq (C_n \wedge S^{nV})_{hO(n)}.$$

where  $S^{nV}$  denotes the one-point compactification of the vector space  $V \otimes \mathbb{R}^n$ .

The functor that we care about is, of course,  $G(V) = \mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V)$ . We will use the notation  $P_n \mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V)$  and  $D_n \mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V)$  to denote its Taylor approximations and homogeneous layers (in the orthogonal calculus sense). The formality of the little cubes operad translates into saying that the orthogonal tower of this functor splits as a product of its layers. Let us outline the main steps that lead to this conclusion. To begin with, each layer of the functor  $\Sigma^\infty C(k, V)$  is ‘‘spherical’’ (a wedge of spheres of the same dimension). Therefore rationally it is an Eilenberg - Mac Lane spectrum. It follows that for the functor  $\mathbf{H}\mathbb{Q} \wedge C(k, V)$  the Taylor tower (in the orthogonal calculus sense) coincides, up to a regrading, with the Postnikov tower. Also, it splits rationally as a product of Eilenberg - Mac Lane spectra.

**Proposition 0.4.** *For all vector spaces  $V$  of dimension greater than 1 there is a homotopy equivalence*

$$P_n \mathbf{H}\mathbb{Q} \wedge C(k, V) \simeq \prod_{i=1}^n D_i \mathbf{H}\mathbb{Q} \wedge C(k, V)$$

This is all well known. The real point is that because of Theorem 0.2 this splitting is functorial with respect to the maps in the diagram in the middle part of (2). It follows that the diagram itself splits as a product of layer diagrams, and so the homotopy inverse of the diagram splits too. This implies our main theorem

**Theorem 0.5.** *For all vector spaces  $V$  of dimension greater than 1 there is a homotopy equivalence, natural in  $M$ ,*

$$P_n \mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V) \simeq \prod_{i=1}^n D_i \mathbf{H}\mathbb{Q} \wedge \overline{\text{Emb}}(M, V).$$

There is an explicit description of the layers  $D_n \Sigma^\infty \overline{\text{Emb}}(M, V)$  as well as  $D_n \overline{\text{Emb}}(M, V)$  as the twisted cohomology of certain spaces of trees and graphs ‘‘grafted’’ onto  $M$ . In the case of knots they specialize to familiar constructions from knot theory, such as chord diagrams etc. It follows from this description that the homotopy groups of layers depend only on the homology of  $M$  (this is true integrally). Combining this with Theorem 0.5, we obtain the following theorem

**Theorem 0.6.** *Suppose that  $2 \dim(M) + 1 < \dim V$ . Then the rational homology of  $\overline{\text{Emb}}(M, V)$  depends only on the rational homology of  $M$ .*

Finally, we show that a similar analysis can be performed for  $\overline{\text{Emb}}(M, V)$  using the coformality of the little cubes operad. Consider the two Taylor towers of  $\overline{\text{Emb}}(M, V)$  provided by embedding calculus and orthogonal calculus. These give rise to two spectral sequences converging to the homotopy groups of  $\overline{\text{Emb}}(M, V)$  if  $\dim(V) - \dim(M) > 3$ . In the following theorem and corollary all the spaces are rationalized

**Theorem 0.7.** *The first of the two aforementioned spectral sequences collapses at  $E_2$  and the second one collapses at  $E_1$ .*

**Corollary 0.8.** *If  $\dim(V) - \dim(M) > 3$  then  $\pi_*(\overline{\text{Emb}}(M, V)_{\mathbb{Q}})$  depends only on the rational homology of  $M$  for  $* > 0$ .*

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# Stable homotopy theory and the derived category of $\mathbb{P}^1$

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This is a report on joint work with Stefan Schwede, summarising my talk given on the Skye topology conference 2005. More details and proofs will be given in a separate paper.

## Spectra and $S$ -modules

A *spectrum* is a sequence  $X = \{X_0, X_1, \dots\}$  of pointed spaces (*i.e.*, simplicial sets), equipped with structure maps  $\lambda_i: S^1 \wedge X_i \longrightarrow X_{1+i}$ . (Here  $S^1 = \Delta^1 / \partial\Delta^1$  is the standard simplicial 1-sphere.) The homotopy groups  $\pi_k(X)$  of  $X$  are defined as the colimit of the sequence

$$\pi_k(X_0) \xrightarrow{S^1 \wedge \cdot} \pi_{k+1}(S^1 \wedge X_0) \xrightarrow{\pi_k(\lambda_0)} \pi_{k+1}(X_1) \xrightarrow{S^1 \wedge \cdot} \dots$$

(if  $k < 0$  the first few terms of this sequence are not defined, but eventually it will be a sequence of ABELian groups).

A more conceptual approach is the following. We consider the sequence  $X = \{X_0, X_1, \dots\}$  as a *graded space*. A particularly important example is the graded space  $S := \{S^0, S^1, S^2, \dots\}$ , the sequence of spheres (where we define inductively  $S^n := S^1 \wedge S^{n-1}$ ).

If  $X$  and  $Y$  are graded spaces, their *tensor product* is the graded space  $X \otimes Y$  defined by

$$(X \otimes Y)_n = \bigvee_{p+q=n} X_p \wedge Y_q.$$

The tensor product of graded spaces is symmetric monoidal with unit object  $S^0 = \{S^0, *, *, \dots\}$ .

It is easily seen that the associativity isomorphisms  $S^j \wedge S^k \cong S^{j+k}$  yield a map  $S \otimes S \longrightarrow S$  of graded spaces. In fact, they equip  $S$  with the structure of a monoid object in the category of graded spaces.

A *left  $S$ -module* is a graded space  $X$  together with an action  $S \otimes X \longrightarrow X$  of  $S$  on  $X$  (which satisfies the usual associativity and unitality conditions). Since  $S$  is the free associative monoid generated by  $S^1$  in degree 1, the category of left  $S$ -modules is equivalent to the category of spectra.

Similarly, we can define a category of right  $S$ -modules which is equivalent to the category of “spectra” with suspensions acting from the right.

## Homotopy groups of $S$ -modules

We can now give a reformulation of the definition of homotopy groups. Given a left  $S$ -module  $X$ , define a graded abelian group  $\Pi_k(X)$ ,  $k \in \mathbb{Z}$ , by

$$\Pi_k(X) := \bigoplus_{j \geq 2-k} \pi_{j+k}(X_j) .$$

The structure maps of  $X$  (considered as a spectrum) induce a degree 1 self-map of  $\Pi_k(X)$ . Hence we can consider  $\Pi_k(X)$  as a graded  $\mathbb{Z}[L]$ -module where  $L$  is an indeterminate. The homotopy group  $\pi_k(X)$  is naturally isomorphic to the degree 0 part of the localised graded module  $\Pi_k(X)_{(L)} = \Pi_k(X) \otimes_{\mathbb{Z}[L]} \mathbb{Z}[L, L^{-1}]$ .

## $S$ -Bimodules and their homotopy sheaves

An  $S$ -bimodule is a graded space  $X$  equipped with compatible structures  $S \otimes X \otimes S \longrightarrow X$  as left and right  $S$ -module. Explicitly, an  $S$ -bimodule is a collection of pointed simplicial sets  $X = \{X_0, X_1, X_2, \dots\}$  and structure maps

$$\lambda_n: S^1 \wedge X_n \longrightarrow X_{1+n} \quad \text{and} \quad \rho_n: X_n \wedge S^1 \longrightarrow X_{n+1}$$

such that the following diagram commutes for all  $n \in \mathbb{N}$ :

$$\begin{array}{ccc} S^1 \wedge X_n \wedge S^1 & \xrightarrow{\text{id} \wedge \rho_n} & S^1 \wedge X_{n+1} \\ \lambda_n \wedge \text{id} \downarrow & & \downarrow \lambda_{n+1} \\ X_{1+n} \wedge S^1 & \xrightarrow{\rho_{1+n}} & X_{1+n+1} \end{array}$$

Informally, an  $S$ -bimodule consists of two spectra with the same underlying graded space and compatible structure maps.

Given an  $S$ -bimodule  $X$ , the graded abelian group  $\Pi_k(X)$  has the structure of a graded  $\mathbb{Z}[L, R]$ -module; the action of the indeterminates  $L$  and  $R$  are induced by left and right structure maps  $\lambda_n$  and  $\rho_n$ , respectively. We denote by  $\tilde{\pi}_k(X)$  the associated quasi-coherent sheaf on  $\mathbb{P}_{\mathbb{Z}}^1 = \text{Proj}(\mathbb{Z}[L, R])$ .

## Stable model structures

The projective line has four interesting open subsets: The projective line itself, two affine lines  $U_L := \{R \neq 0\}$  and  $U_R := \{L \neq 0\}$ , and the algebraic torus  $U_0 := U_L \cap U_R$ . We call these sets the *distinguished open subsets*.

Let  $U$  be a distinguished open set. A map  $f: X \longrightarrow Y$  of  $S$ -bimodules is called a  $U$ -equivalence if for all  $n \in \mathbb{Z}$  the induced map of restrictions of sheaves

$$\pi_n(f)|_U: \pi_n(X)|_U \longrightarrow \pi_n(Y)|_U$$

is an isomorphism. Since the homotopy sheaves are quasi-coherent, we can characterise equivalences for  $U$  affine by their effect on sections over  $U$ : The map  $f: X \longrightarrow Y$  is a  $U$ -equivalence if and only if it induces isomorphisms of modules  $\Gamma(U, \tilde{\pi}_n(X)) \longrightarrow \Gamma(U, \tilde{\pi}_n(Y))$  for all  $n \in \mathbb{Z}$ . Moreover,  $f$  is a  $\mathbb{P}^1$ -equivalence if and only if it is a  $U_L$ -equivalence as well as a  $U_R$ -equivalence.

**1 Theorem.** *Let  $U$  be a distinguished open subset of  $\mathbb{P}^1$ . The category of  $S$ -bimodules admits a simplicial closed model structure where a map  $f$  is a weak equivalence if and only if it is a  $U$ -equivalence. The model structure is stable in the sense that simplicial suspension and loop functors induce mutually inverse equivalences on the homotopy category.*

The idea here is that an  $S$ -bimodule is a homotopical analogue of a quasi-coherent sheaf on  $\mathbb{P}^1$ , and the  $U$ -equivalences capture exactly the homotopical information living over the subset  $U$ .

The four model structures of the theorem are interrelated: If  $U \supseteq V$  are distinguished open subsets of  $\mathbb{P}^1$ , the identity functor is a left Quillen functor from the  $U$ -model structure on the category of  $S$ -bimodules to the  $V$ -structure. Moreover, morphisms in the  $\mathbb{P}^1$ -homotopy category of  $S$ -bimodules are determined by morphisms in the  $U$ -homotopy categories with  $U$  ranging over the affine distinguished sets. In other words, we have constructed a “bundle of homotopy categories” on  $\mathbb{P}^1$ , and morphisms are determined by local data:

**2 Theorem.** *Let  $X$  and  $Z$  be  $S$ -bimodules. Let  $(\underline{S\text{-mod-}S})_U(X, Z)$  denote a space having the homotopy type of a homotopy function complex of maps from  $X$  to  $Z$  in the  $U$ -model structure of  $S$ -bimodules. One can choose these spaces in such a way that there results a commutative diagram which is homotopy cartesian:*

$$\begin{array}{ccc}
 (\underline{S\text{-mod-}S})_{\mathbb{P}^1}(X, Z) & \longrightarrow & (\underline{S\text{-mod-}S})_{U_R}(X, Z) \\
 \downarrow & & \downarrow \\
 (\underline{S\text{-mod-}S})_{U_L}(X, Z) & \longrightarrow & (\underline{S\text{-mod-}S})_{U_0}(X, Z)
 \end{array} \tag{*}$$

*In particular, the associated long exact Mayer-Vietoris sequence contains a description of  $\mathrm{Ho}_{\mathbb{P}^1}(X, Y) = \pi_0(\underline{S\text{-mod-}S})_{\mathbb{P}^1}(X, Z)$  in terms of  $\mathrm{Ho}_{U_L}(X, Y)$ ,  $\mathrm{Ho}_{U_R}(X, Y)$  and  $\pi_1(\underline{S\text{-mod-}S})_{U_0}(X, Z)$ .*

## Abelian spectra

The above can be re-done in a linearised setting. Let  $A$  denote a noetherian commutative ring (with unit). An  $A$ -abelian spectrum is a graded simplicial  $A$ -module  $X = \{X_0, X_1, \dots\}$  together with structure maps (maps of simplicial sets)  $\lambda_n: S^1 \wedge X_n \longrightarrow X_{n+1}$  such that the adjoint maps  $X_n \longrightarrow \Omega X_{n+1}$

are homomorphisms of simplicial  $A$ -modules. Equivalently, we can proscribe structure maps  $\tilde{A}[S^1] \otimes_A X_n \longrightarrow X_{n+1}$  which are homomorphisms of simplicial  $A$ -modules (where  $\tilde{A}[S^1] = A[S^1]/A[*]$  is the reduce free simplicial  $A$ -module generated by  $S^1$ , and tensor means level-wise tensor product).

As before, we can give a re-interpretation using graded objects. The category of graded simplicial  $A$ -modules has a symmetric monoidal product given by

$$(X \otimes_A Y)_n = \bigoplus_{i+j=n} X_i \otimes_A Y_j .$$

The graded simplicial  $A$ -module

$$\tilde{A}[S] = \{\tilde{A}[S^0], \tilde{A}[S^1], \dots\}$$

is a monoid object with respect to the tensor product; the structure maps are induced by the maps  $\tilde{A}[S^i] \otimes_A \tilde{A}[S^j] \longrightarrow \tilde{A}[S^{i+j}]$  given by concatenation of generators; they can also be described as the linearisations of the isomorphisms  $S^i \wedge S^j \longrightarrow S^{i+j}$ . An abelian spectrum is then nothing but a left  $\tilde{A}[S]$ -module, that is, a graded simplicial  $A$ -module equipped with a left action of  $\tilde{A}[S]$ .

It is well known that the category of abelian spectra admits a stable model structure where a map is a weak equivalence (or fibration) if and only if it is a weak equivalence (or fibration) of underlying (non-abelian) spectra. Moreover, the homotopy category of abelian spectra is equivalent to the (unbounded) derived category of  $A$ .

### Abelian $S$ -bimodules

An  $A$ -abelian  $S$ -bimodule is a graded simplicial  $A$ -module equipped with compatible structures as a left and right  $\tilde{A}[S]$ -module. Each abelian  $S$ -bimodule  $X$  has an underlying  $S$ -bimodule (forget the linear structure), so we know what homotopy sheaves  $\tilde{\pi}_n(X)$  and  $U$ -equivalences should be. Note that now  $\tilde{\pi}_n(X)$  is naturally a quasi-coherent sheaf on  $\mathbb{P}_A^1$ , the projective line over  $A$ .

**3 Theorem.** *Let  $U$  be a distinguished open subset of  $\mathbb{P}^1$ . The category of  $A$ -abelian  $S$ -bimodules admits a simplicial closed model structure where a map is a weak equivalence if and only if it is a  $U$ -equivalence. The model structure is stable in the sense that simplicial suspension and loop functors induce mutually inverse equivalences on the homotopy category.*

As in the case of  $S$ -bimodules the associated homotopy categories for the various  $U$  assemble to a “bundle of homotopy categories”; Theorem 2 remains valid *mutatis mutandis*. This enables us to identify the homotopy categories of abelian  $S$ -bimodules with well-known objects in algebra:

**4 Theorem.** *Let  $U$  be a distinguished open subset of  $\mathbb{P}_A^1$ . The homotopy category of  $A$ -abelian  $S$ -bimodules with respect to  $U$ -equivalences is equivalent, as*

*a triangulated category, to the (unbounded) derived category of quasi-coherent sheaves on  $U$ . If  $U$  is affine ( $U \neq \mathbb{P}_A^1$ ), the equivalence can be realised by a Quillen equivalence of model categories.*

# ON THE SIGNATURE OF STRING MANIFOLDS

MICHAEL JOACHIM

ABSTRACT. We show a generalization of a theorem of Ochanine and determine (up to a factor of 2) the image of the signature homomorphism regarded as a ring homomorphism from the string cobordism group to the integers.

By definition a closed manifold  $M$  has a string structure if and only if its classifying map  $M \rightarrow BO$  has a lift to the 7-connected cover  $BO\langle 8 \rangle$ . Because of the lift the Pontryagin numbers of such a manifold satisfy some higher divisibility conditions which should also have an effect on the possible values the signature of a closed string manifold can attain. A corresponding phenomenon has already been observed for the signature of spin manifolds. Namely Ochanine has shown in [4] that the signature of an  $8k+4$ -dimensional closed spin manifold is divisible by 16, extending the celebrated result of Rochlin for 4-manifolds. On the other hand, in the spin case we have a lot of good examples which realize minimal non-trivial absolute values for the signature. The  $K3$ -surface of dimension 4 has signature  $\sigma(K3) = -16$  and the quaternionic projective spaces  $\mathbb{H}P^{2n}$  have signature  $\sigma(\mathbb{H}P^{2n}) = 1$ . Taking all quaternionic projective spaces as well the products with those and  $K3$  we can use Ochanines theorem to determine the image of signature homomorphism from the spin bordism ring to the integers:

$$\Omega_{8k}^{spin} = \mathbb{Z}, \quad \Omega_{8k+4}^{spin} = 16\mathbb{Z}.$$

We essentially want to follow the same line of argument in the case of string manifolds. First we show a variant of Ochanines theorem, and then we look for suitable examples which realize minimal non-trivial absolute values of the signature. One essential problem we have to face is that none of the above examples carries a string structure. However there is one famous example of a string manifold of signature 1, the Cayley plane  $CaP$ . It can be thought of as the 2-dimensional projective space over the Cayley numbers, but there are no higher dimensional analogues to this. Nevertheless we can take products of Cayley planes to show that  $\sigma(\Omega_{16k}^{string}) = \mathbb{Z}$ , so we may concentrate on dimensions which are not a multiple of 16.

It is well-known that in dimensions less than 16 we have

$$\sigma(\Omega_4^{string}) = 0; \quad \sigma(\Omega_8^{string}) = 32 \cdot 7 \cdot \mathbb{Z}; \quad \sigma(\Omega_{12}^{string}) = 256 \cdot 31 \cdot \mathbb{Z}.$$

The values in these dimensions can all be realized by so-called almost-parallelizable manifolds (cf. [1, page 91]). Our result for the higher dimensions is as follows.

**Theorem A.** *The image of the signature homomorphism in dimensions  $i \geq 16$  for  $i \equiv 0 \pmod{4}$  is given by*

$$\begin{aligned} \sigma(\Omega_{16k}^{string}) &= \mathbb{Z}; & \sigma(\Omega_{16k+4}^{string}) &= 16\mathbb{Z} \\ \sigma(\Omega_{16k+8}^{string}) &= 8\mathbb{Z}; & \sigma(\Omega_{16k+12}^{string}) &= 16\mathbb{Z} \text{ or } 32\mathbb{Z}. \end{aligned}$$

Note that Ochanines result implies that  $16\mathbb{Z}$  is the biggest possible image in dimensions which are  $4 \pmod 8$ . That  $8\mathbb{Z}$  is the biggest possible image in dimensions which are  $8 \pmod 16$  follows rather easily from the following lemma.

**Lemma B.** *Let  $M$  be a  $16k+8$ -dimensional closed manifold. For  $x \in H^{8k+4}(M; \mathbb{Z}/2)$  we have*

$$Sq^{8k+4}x = v_4(M)Sq^{8k}x + v_6(M)Sq^{8k-2}x + v_7(M)Sq^{8k-3}x,$$

where the  $v_i(M)$  denote the Wu classes of  $M$ . In particular, if  $M$  is a string manifold we have  $Sq^{8k+4} = 0$  on  $H^{8k+4}(M; \mathbb{Z}/2)$ .

It then follows that the intersection form of a  $16k + 8$ -dimensional closed string manifold is even, and therefore its signature must be a multiple of 8.

To get the above theorem we now need to provide the relevant examples. To motivate our procedure we would like recall the following theorem.

**Theorem C** (Kreck-Stolz [3, Proposition 3.3]). *Let  $M$  be a closed spin manifold. If the  $KO$ -theoretic index of the Dirac operator on  $M$  vanishes then  $M$  is spin cobordant to the total space of an  $\mathbb{H}P^2$ -bundle.*

Note that the  $KO$ -index map  $\Omega_*^{spin} \rightarrow KO_*$  is injective for  $* < 8$  and that  $\mathbb{H}P^2$  generates the kernel in dimension 8. After Kreck and Stolz proved this theorem the question came up whether or not (2-locally) the kernel of the modified Witten genus  $\Omega_*^{string} \rightarrow tmf_*$  into the ring of topological modular forms  $tmf_*$  is generated by total spaces of Cayley plane bundles. Note that (2-locally) the modified Witten genus is injective for  $* < 16$  and that the Cayley plane generates the kernel in dimension 16. We in fact have dealt with the above question, and work in progress eventually will show that (2-locally) the kernel indeed is generated by Cayley plane bundles throughout a range, say at least up to dimension 32. On the other hand we know by the work of Hopkins and Mahowald [2] that the image of the Witten genus in dimensions 20, 24 and 28 can be generated by the images of string manifolds whose cobordism classes have high 2-Adams filtration. Thus it is natural to look for Cayley plane bundles to realize the minimal non-trivial absolute values for the signature of string manifolds in these dimensions.

Recall that the Cayley plane  $CaP$  can be identified with a homogeneous space, namely with the quotient  $F_4/Spin(9)$ . From this it follows that  $F_4$  acts through isometries on  $CaP$ , and indeed  $F_4$  is the full isometry group of  $CaP$ . Thus Cayley plane bundles over a manifold  $M$  are classified by maps  $M \rightarrow BF_4$ . Given such a map  $f : M \rightarrow BF_4$  let  $p_f : E_f \rightarrow M$  denote the associated Cayley plane bundle. Then  $TE_f \cong p_f^*TM \oplus \Delta_f$ , where  $\Delta_f$  is the tangent bundle along the fiber. Thus in order to have a string structure on the total space  $E_f$  we need to have  $p_f^*p_1(TM) = -p_1(\Delta_f)$ . Since the first Pontryagin class of the tangent bundle along the fiber of the universal Cayley plane bundle over  $BF_4$  is four times the canonical generator of  $H^4(BF_4; \mathbb{Z})$  we need to concentrate on bundles whose base is a closed spin manifold with  $4|p_1(M)$ .

The construction of the 20-dimensional example now is very easy. Consider the  $K3$ -surface. We have  $p_1(K3)[K3] = -48$ . Thus we can take the map  $f : K3 \rightarrow S^4 \rightarrow BF_4$  which is the composition of the collapse map to the top cell followed by 12 times the generator in  $\pi_4(BF_4)$ . It follows that the corresponding total space  $E_f$  is a closed string manifold, and by rigidity of the signature we have

$\sigma(E_f) = \sigma(K3) \cdot \sigma(CaP) = -16$ . The tricky part is constructing the relevant examples in dimensions 24 and 28.

To get the 24-dimensional example we use the so-called plumbing construction, which long ago prominently has been used for constructing almost-parallelizable manifolds (compare [1, 6.5]). First let us take 8 copies of the tangent disk bundle of  $S^4$  and let us plumb them together according to the matrix  $E_8$ . This way one obtains an 8-dimensional manifold  $W_1$  with boundary a homotopy sphere. That the boundary is a homotopy sphere essentially follows from the fact that  $E_8$  is invertible, and the boundary in fact yields a generator of  $\Theta_7$ , the group of 7-dimensional homotopy spheres. Similarly one can use the two-dimensional hyperbolic matrix to plumb together two copies of the disk bundle  $E$  over  $S^4$  with  $\chi(E) = 0$  and  $p_1(E)[S^4] = 4$ . The boundary of the resulting manifold  $W_2$  then also is a homotopy sphere, and we can show that it also yields a generator of  $\Theta_7$ . Thus there is a number  $k$  such that  $W = W_1 \# k \cdot W_2$  bounds the standard sphere. We now can fill in a round disk to get a closed manifold  $M$  with signature 8 and  $p_1(M)$  divisible by 4. This then allows to define a suitable map from the 4-skeleton of  $M$  to  $BF_4$  and extend it to a map  $f : M \rightarrow BF_4$  so that the corresponding total space  $E_f$  has a string structure. Again by rigidity of the signature we have  $\sigma(E_f) = \sigma(M) \cdot \sigma(CaP) = 8$ .

What dimension 28 is concerned, here we only have an example with signature equal to 32. We originally thought that a plumbing construction similarly as above would yield a 28-dimensional closed string manifold with signature 16, but this finally did not work out. We therefore presently don't know whether the image of  $\Omega_{28}^{string}$  is  $16\mathbb{Z}$  or  $32\mathbb{Z}$ . We nevertheless believe that there is a string manifold of dimension  $12 \pmod{16}$  which does have signature 16.

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# CELLULARIZATION OF CLASSIFYING SPACES

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ABSTRACT. In this lecture it is described the  $B\mathbb{Z}/p$ -cellularization of classifying spaces of finite groups. It is shown that it has the homotopy type of the classifying space of a finite  $p$ -group, or else it has an infinite number of nonzero homotopy groups, which in turn are either  $p$ -torsion free or else infinitely many of them contain  $p$ -torsion. By means of techniques related to fusion systems, concrete examples where  $p$ -torsion appears are described, and the cellularization is computed in a number of cases. We also use these results to obtain exotic representations of these groups.

This is a summary of my lecture held at the “Conference in Pure and Applied Topology”, Skye, Tuesday June 21st 2005. The content of the lecture is joint work with J. Scherer, and will hopefully appear in the paper [FS05].

Consider  $B\mathbb{Z}/p$ , the classifying space of the cyclic group of  $p$ -elements. A space  $X$  is called  $B\mathbb{Z}/p$ -cellular if it can be built from  $B\mathbb{Z}/p$  by means of pointed homotopy colimits. There exists a  $B\mathbb{Z}/p$ -cellularization functor  $CW_{B\mathbb{Z}/p}$  that provides the best possible  $B\mathbb{Z}/p$ -cellular approximation: for every space  $X$ , the natural map  $CW_{B\mathbb{Z}/p}X \rightarrow X$  induces a weak equivalence between pointed mapping spaces  $\text{Map}_*(B\mathbb{Z}/p, CW_{B\mathbb{Z}/p}X) \simeq \text{Map}_*(B\mathbb{Z}/p, X)$ .

Our interest lies essentially in using the cellularization functor to study the  $p$ -primary part of the homotopy of the classifying space of a finite group  $G$ . This approach was already suggested by Dror-Farjoun in [Far96, Example 3.C.9], developed by Chachólski ([Cha96]) and successfully exploited in the last years by Bousfield ([Bou97]), Rodríguez-Scherer ([RS01]), and others.

In the present lecture we focus our attention in describing the  $B\mathbb{Z}/p$ -cellularization of  $BG$ , for  $G$  finite. A first attempt to understand  $CW_{B\mathbb{Z}/p}BG$  was undertaken in my previous work [Flo], where in particular the fundamental group of the cellularization

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was described as an extension of the group-theoretical  $\mathbb{Z}/p$ -cellularization of a subgroup of  $G$  by a finite  $p$ -torsion free abelian group. So in the sequel we concentrate on the higher homotopy groups  $\pi_n(CW_{B\mathbb{Z}/p}BG)$ , for  $n \geq 2$ .

Our first result establishes that, in general, the space  $CW_{B\mathbb{Z}/p}BG$  has infinitely many non-trivial homotopy groups. This contrasts with the results in [CCS]: for easier spaces to work with, such as  $H$ -spaces or classifying spaces of nilpotent groups, the assumption that the mapping space  $\text{Map}_*(B\mathbb{Z}/p, X)$  is discrete implies that  $CW_{B\mathbb{Z}/p}X$  is aspherical.

**Proposition 0.1.** *Let  $G$  be a finite group. Then  $CW_{B\mathbb{Z}/p}BG$  is either the classifying space of a finite  $p$ -group, or it has infinitely many non-trivial homotopy groups.*

*Sketch proof.* The cellularization  $CW_{B\mathbb{Z}/p}BG$  is the homotopy fibre of Chachólski's map  $BG \rightarrow P_{\Sigma B\mathbb{Z}/p}C$  ([Cha96, 20.10]); here  $C$  stands for the homotopy cofibre of the evaluation map  $\bigvee B\mathbb{Z}/p \rightarrow BG$ , where the wedge is extended to all the homotopy classes of maps  $B\mathbb{Z}/p \rightarrow BG$ . It is shown that if  $G$  is not a  $p$ -group,  $P_{\Sigma B\mathbb{Z}/p}C$  breaks as a product  $\prod BG_q^\wedge \times (P_{\Sigma B\mathbb{Z}/p}C)_p^\wedge$  for every  $q \mid |G|$  such that  $q \neq p$ , and then one can conclude by Levi's dichotomy theorem ([Lev95, Theorem 1.1.4]). Here and afterwards,  $X_p^\wedge$  stands for Bousfield-Kan  $p$ -completion of  $X$  (see [BK72] for details).  $\square$

It is worth to remark here that the homotopy groups of a  $B\mathbb{Z}/p$ -cellular space are not necessarily  $p$ -groups, so we wish to understand not only the  $p$ -primary part, but also the more accessible  $p'$ -primary part. The latter is indeed closely related to the  $B\mathbb{Z}/p$ -nullification of  $BG$ , due to Chachólski's description ([Cha96, 20.10]) of the cellularization as a small variation of the homotopy fiber of the  $B\mathbb{Z}/p$ -nullification map. In [Flo, 3.5] the  $B\mathbb{Z}/p$ -nullification  $P_{B\mathbb{Z}/p}BG$  was already identified, when  $G$  is generated by elements of order  $p$ , as the product (taken over all primes  $q$  different from  $p$ ) of the  $q$ -completions  $BG_q^\wedge$ .

**Proposition 0.2.** *Let  $G$  be a finite group generated by elements of order  $p$ . Then either the cellularization  $CW_{B\mathbb{Z}/p}BG$  has infinitely many homotopy groups containing  $p$ -torsion or it fits in a fibration*

$$CW_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow \prod_{q \neq p} BG_q^\wedge,$$

where the (finite) product is taken over all primes  $q$  dividing the order of  $G$ , and the right map is the product of the completions.

*Sketch proof.* According to 0.1, it is enough to show that if  $P_{\Sigma B\mathbb{Z}/p}C$  is nontrivial, it has an infinite number of nonzero homotopy groups. This is proved for every simply-connected torsion  $\Sigma B\mathbb{Z}/p$ -local space by analyzing a convenient Postnikov fibration. The reduction to groups generated by order  $p$  elements, that is necessary to guarantee the 1-connectedness of  $P_{\Sigma B\mathbb{Z}/p}C$ , is actually not crucial.  $\square$

Note that in the second case the higher homotopy groups of the cellularization are those of  $\Omega(\prod_{q \neq p} BG_q^\wedge)$ . In particular, if  $G$  is generated by elements of order  $p$ ,  $CW_{B\mathbb{Z}/p}BG$  coincides with the homotopy fiber of the nullification map  $\bar{P}_{B\mathbb{Z}/p}$  unless there appears (a lot of)  $p$ -torsion in the higher homotopy groups. The main question which remains unanswered up to this point is thus to determine if there actually exist groups for which  $CW_{B\mathbb{Z}/p}BG$  does contain  $p$ -torsion in its higher homotopy groups! This turns out to depend heavily on the  $p$ -complete classifying space  $BG_p^\wedge$ .

**Proposition 0.3.** *Let  $G$  be a group generated by order  $p$  elements which is not a  $p$ -group. Then the universal cover of  $CW_{B\mathbb{Z}/p}BG$  is  $p$ -torsion free if and only if the  $p$ -completion of  $BG$  is  $B\mathbb{Z}/p$ -cellular.*

*Sketch proof.* By decomposing the homotopy cofibre of the evaluation  $\bigvee B\mathbb{Z}/p \rightarrow BG$  as a product of its  $q$ -completions, we are able to show that  $(CW_{B\mathbb{Z}/p}BG)_p^\wedge$  is homotopy equivalent to  $CW_{B\mathbb{Z}/p}(BG_p^\wedge)$ . Then, as the  $p$ -completion of  $BG$  is  $B\mathbb{Z}/p$ -cellular if and only if  $(CW_{B\mathbb{Z}/p}BG)_p^\wedge$  is so, the statement is an easy consequence of Chachólski's fibration ([Cha96, 20.10]) referred to  $BG_p^\wedge$ .  $\square$

**Remark 0.4.** *It would be really interesting to identify a wider class of non-simply connected spaces  $X$  for which  $(CW_{B\mathbb{Z}/p}X)_p^\wedge \simeq CW_{B\mathbb{Z}/p}(X_p^\wedge)$*

We notice that the presence of  $p$ -torsion in the higher homotopy groups of the cellularization of  $BG$  depends on the fusion properties of  $G$ . If  $S$  is a Sylow  $p$ -subgroup, define  $Cl(S)$  as the smallest *strongly closed* subgroup of  $S$  containing all order  $p$  elements of  $S$ . This means that whenever a conjugate  $g x g^{-1}$  of an element  $x \in Cl(S)$  by an element  $g \in G$  lies in  $S$ , then in fact  $g x g^{-1} \in Cl(S)$ . Our next result shifts the problem to the study of fusion systems.

**Proposition 0.5.** *Let  $G$  be a group generated by elements of order  $p$ . If  $S$  is equal to  $Cl(S)$ , then the universal cover of  $CW_{B\mathbb{Z}/p}BG$  is  $p$ -torsion free.*

*Sketch proof.* We first show that the statement is equivalent to show that every map  $f : BG \rightarrow Z$  to a simply-connected,  $H\mathbb{Z}/p$ -local  $\Sigma B\mathbb{Z}/p$ -null space  $Z$  that is inessential when precomposing with any map  $B\mathbb{Z}/p \rightarrow BG$  is itself inessential. According to a theorem of Dwyer ([Dwy96, Theorem 1.4]), this is equivalent to show that  $f$  is null-homotopic when restricted to the classifying space of every  $p$ -Sylow subgroup. In our case  $S = Cl(S)$ , and the latter can be constructed using an ascending chain  $\Omega_1(G) = Cl_0(S) \leq Cl_1(S) \dots \leq Cl_n(S) = Cl(S)$ , where  $\Omega_1(S)$  is the subgroup of  $S$  generated by the order  $p$  elements (the  $\mathbb{Z}/p$ -socle). Now,  $f|_{\Omega_1(S)}$  is inessential, because  $f$  is so when restricted to order  $p$  subgroups. The proof finishes by induction, using Zabrodsky's Lemma ([Dwy96, Proposition 3.5]) in the induction step.  $\square$

There are very few groups containing a proper strongly closed subgroup. This yields many examples of  $B\mathbb{Z}/p$ -cellular spaces whose higher homotopy is  $p$ -torsion free, including the cellularization of classifying spaces of the symmetric groups. It is a delicate problem to obtain an explicit description of the  $p$ -primary part of  $CW_{B\mathbb{Z}/p}BG$ . For this purpose we focus on groups in which the normalizer of a Sylow  $p$ -subgroup controls the fusion. Recall that if  $G$  is a finite group, the normalizer  $N_G(S)$  of the Sylow  $p$ -subgroup  $S$  controls fusion if, whenever  $P < G$  is a  $p$ -subgroup and  $gPg^{-1} < N_G(S)$ , we have that  $g = hc$ , with  $h \in N_G(S)$  and  $c \in C_G(P)$ , the centralizer of  $P$  in  $G$ .

**Proposition 0.6.** *Let  $G$  be a finite group generated by elements of order  $p$  and  $S$  a Sylow  $p$ -subgroup. Assume that the normalizer  $N_G(S)$  controls the fusion in  $G$ . Then  $CW_{B\mathbb{Z}/p}BG$  fits in a fibration*

$$CW_{B\mathbb{Z}/p}BG \longrightarrow BG \longrightarrow B\Gamma_p^\wedge \times \prod_{q \neq p} BG_q^\wedge,$$

where  $\Gamma = N_G(S)/Cl(S)$ .

*Sketch proof.* According again to ([Cha96, 20.10]), it is enough to show that  $B\Gamma_p^\wedge$  is the  $\Sigma B\mathbb{Z}/p$ -nullification of the cofibre  $C$  of the evaluation  $\vee B\mathbb{Z}/p \rightarrow BG$ . This is done by showing the existence of a map  $C \rightarrow B\Gamma_p^\wedge$  which factors through  $P_{\Sigma B\mathbb{Z}/p}C$ , and then proving that it has a homotopy inverse, that comes from applying again

Zabrodsky's lemma to the fibration

$$BCl(S) \rightarrow BN_G(S) \rightarrow B\Gamma.$$

□

At the prime 2, Foote actually characterized in [Foo97] the groups containing a proper strongly closed subgroup. Hence, the previous proposition allows to identify the only two families of simple groups  $G$  for which 2-torsion appears in the higher homotopy groups of  $CW_{B\mathbb{Z}/2}BG$ . For example, there is a fibration

$$CW_{B\mathbb{Z}/2}BSz(2^n) \rightarrow BSz(2^n) \rightarrow B((\mathbb{Z}/2)^n \rtimes \mathbb{Z}/(2^n - 1))_2^\wedge \times \prod_{q \neq 2} BSz(2^n)_q^\wedge$$

and a similar description holds for the groups  $U_3(2^n)$ . At odd primes a similar computation can be done for certain projective special linear groups, but we do not claim that they are the unique simple groups for which  $p$ -torsion appears.

We finish by taking profit of the previous study of the 2-local structure of the Suzuki groups to obtain unitary representations of the classifying spaces that are trivial when precomposing with maps coming from  $B\mathbb{Z}/2$ . To our knowledge, these are the first known representations with that property.

**Proposition 0.7.** *There exists a non-trivial map  $BSz(2^n) \rightarrow BU(2^n - 1)_2^\wedge$  such that the composition  $B\mathbb{Z}/2 \rightarrow BSz(2^n) \rightarrow BU(2^n - 1)_2^\wedge$  is null-homotopic for every cyclic subgroup  $\mathbb{Z}/2$  in  $Sz(2^n)$ .*

Interesting representations can also be obtained for the groups  $U_3(2^n)$  in the prime 2, and for projective special linear groups  $PSL_2(q)$  in certain odd primes.

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# Homology operations in string topology

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This note is part of some joint work with Kate Gruher. Chas and Sullivan constructed in [3] new operations on the free loop space homology of a manifold. This started a stream of ideas that form now a subject called ‘string topology’, from the title of the fundamental paper by Chas and Sullivan. The construction of the main operation, the loop product, mixes the intersection product on the homology of the manifold with the Pontrjagin product of its based loop space. We observe that the construction can be extended in many directions.

First of all, the product can be constructed on the homology of the total space of any fiberwise monoid over a closed manifold. This is a fibration such that each fiber is a monoid and the product varies continuously.

Second, suppose that the fibers have a finer multiplicative structure so that we have a fiberwise iterated loop spaces (or a fiberwise algebra over the little cubes operad) . Then the total space of the fibration admits homology operations similar to the Dyer-Lashof-Cohen operations of iterated loop spaces. For single loop spaces compare [7, 8]. These new operations combine in a sense the operations on the fibers and the Poincare duals of the Steenrod operations on the base.

Third, given a fiberwise monoid  $\eta$ , consider a fibration  $\alpha$  acted on fiberwise by  $\eta$ : each fiber of  $\alpha$  is acted on by the corresponding monoid fiber of  $\eta$ . Then the total space homology of  $\alpha$  is a module over the total space homology of  $\eta$ .

Some important examples are the following:

0) Mapping spaces:

The evaluation fibration

$$\text{map}_*(S^n, M) \rightarrow \text{map}(S^n, M) \rightarrow M$$

is a fiberwise  $n$ -fold loop space. For any closed  $n$ -manifold  $N$  the fibration  $\text{map}_*(N, M) \rightarrow \text{map}(N, M) \rightarrow M$  admits a fiberwise action by the fiberwise  $n$ -fold loop space above.

1) Holomorphic maps:

Let  $G/P$  denote a homogeneous complex projective manifold. Let  $\text{hol}(G/P)$  and  $\text{hol}_*(G/P)$  be respectively the spaces of free and based holomorphic maps from the Riemann sphere to  $G/P$ . Then the fibration  $\text{hol}_*(G/P) \rightarrow \text{hol}(G/P) \rightarrow$

$G/P$  has a fiberwise action of the little 2-cubes operad [6]. The action on fibers is outlined in [1]. The inclusion into the continuous fibration  $map_*(S^2, G/P) \rightarrow map(S^2, G/P) \rightarrow G/P$  is compatible with the operad action.

2) Gauge theory:

Let  $\nu_k(N)$  be a principal  $G$ -bundle over a closed 4-manifold  $N$ , with  $G$  a simple Lie group and  $k$  the second Chern number. Let  $\mathcal{G}_k(N)$  be the *gauge group* of  $G$ -equivariant automorphisms of the bundle and  $\mathcal{G}_{*k}(N)$  the subgroup of elements fixing a fiber. Then the fibration  $\coprod_k B\mathcal{G}_{*k}(S^4) \rightarrow \coprod_k B\mathcal{G}_k(S^4) \rightarrow BG$  has a fiberwise action of the little 4-cubes operad. Moreover for any closed 4-manifold  $N$  the fibration  $\coprod_k B\mathcal{G}_{*k}(N) \rightarrow \coprod_k B\mathcal{G}_k(N) \rightarrow BG$  is acted on by the fibration of the 4-sphere.

3) Spaces of embeddings:

Let  $K$  be the space of long knots in  $\mathbb{R}^3$  and  $E$  the space of knots in  $S^3$ . Then there is a fibration  $K \rightarrow E \rightarrow SO(4)/SO(2)$  with a fiberwise action of the little 2-cubes operad (up to homotopy). The action on the fibers was discovered by Budney [2].

4) Maps from Riemann surfaces:

The fibration

$$\coprod_g map_*(S_g, M) \rightarrow \coprod_g map(S_g, M) \rightarrow M,$$

where  $S_g$  is a genus  $g$  Riemann surface, is a fiberwise monoid (up to homotopy).

5) *Universal example:*

Let  $Y$  be a monoid with an action of a group (or a monoid with homotopy inverse)  $G$  that preserves the sum, and  $G \rightarrow E \rightarrow M$  a  $G$ -principal fibration over a closed manifold  $M$ . Then  $Y \rightarrow Y \times_G E \rightarrow M$  is a fiberwise monoid. The example is universal because any fiberwise monoid has this form with  $G = \Omega M$  the based loop space and  $E = PM$  the contractible path space. However this helps to construct examples.

Some further comments are in order:

a) there is no module structure in the holomorphic category as opposite to the continuous: the space of holomorphic based maps  $hol_*(\Sigma, G/P)$  from a Riemann surface  $\Sigma$  to  $G/P$  is not a module over  $hol_*(G/P)$ .

b) In example 2) the base is infinite-dimensional and we must pullback the fibration to a finite dimensional skeleton of  $BG$  in order to build a product on the total space.

c) Budney constructed a little disc action on a space homotopy equivalent to the long knots space, and this is the reason why example 3) works up to homotopy.

d) The ‘string’ product can also be constructed at the level of spectra (compare [6], [4] ) and thus for a suitable generalized homology theory.

The new operations allow homology computations that had been out of reach. Namely the product construction goes through in the Serre spectral sequence, as shown by Cohen-Jones, Yan [5], making it into a spectral sequence of algebras. Kallel and I used this feature in [6] to compute the homology of  $hol(\mathbb{C}\mathbb{P}^n)$  and  $map(S^2, \mathbb{C}\mathbb{P}^n)$ . These spaces have an infinite number of components that are not homology equivalent. However we used the module structure to get a periodicity result about the homology of  $map(\Sigma, \mathbb{C}\mathbb{P}^n)$  when  $\Sigma$  is a Riemann surface: the homology mod  $p$  depends only on the residue mod  $p$  of the component degree.

The Serre spectral sequence of a fibration of iterated loop spaces has also operations satisfying a transgression theorem a la Kudo. This allows the computation of the homology of  $B\mathcal{G}_1(S^4)$ . The same technique shows that there is no periodicity pattern in this case, and the homology mod  $p$  of  $B\mathcal{G}_k(N)$  gets more complicated as the  $p$ -exponent of  $k$  grows.

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# ON THE ORBIT SPACE OF A FUSION SYSTEM

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Some of the most prominent conjectures in modular representation theory admit interpretations in cohomological terms of functors on suitable orbit spaces. Section I sketches background material on functor cohomology, Section II recalls the definition and basic properties of  $EI$ -categories, Section III reviews the general concept of subdivisions of  $EI$ -categories of which orbit spaces are particular cases, Section IV contains definitions and properties on fusion systems as needed in Section V to extend contractibility results for orbit spaces to the corresponding spaces associated with arbitrary fusion systems, and Section VI illustrates finally how this is related to Alperin's weight conjecture and its subsequent reformulations and refinements.

## I. FUNCTOR COHOMOLOGY

Let  $\mathcal{C}$  be a small category and let  $k$  be a commutative ring. Denote by  $\hat{\mathcal{C}}$  the category of covariant functors from  $\mathcal{C}$  to  $\text{Mod}(k)$ , with natural transformations of functors as morphisms. The category  $\hat{\mathcal{C}}$  is an abelian category with enough projectives. We denote by  $\mathbf{k}$  the constant covariant functor in  $\hat{\mathcal{C}}$  mapping every object in  $\mathcal{C}$  to  $k$ . The limit  $\varprojlim$  over the category  $\mathcal{C}$  is a covariant functor from  $\hat{\mathcal{C}}$  to  $\text{Mod}(k)$  which is well-known to be isomorphic to the functor  $\text{Hom}_{\hat{\mathcal{C}}}(\mathbf{k}, -)$ . For any covariant functor  $\mathcal{A} : \mathcal{C} \rightarrow \text{Mod}(k)$  we set

$$H^*(\mathcal{C}; \mathcal{A}) = \text{Ext}_{\hat{\mathcal{C}}}(\mathbf{k}; \mathcal{A}) \cong \varprojlim^*(\mathcal{A}) .$$

Explicitly, if  $\mathcal{P}$  is a projective resolution of  $\mathbf{k}$  in  $\hat{\mathcal{C}}$  then

$$H^n(\mathcal{C}; \mathcal{A}) = H^n(\text{Hom}_{\hat{\mathcal{C}}}(\mathcal{P}; \mathcal{A}))$$

for any integer  $n \geq 0$ . There is a canonical projective resolution of  $\mathbf{k}$  in  $\hat{\mathcal{C}}$ , analogous to the bar resolution for a finite group. Base change techniques are amongst the most fundamental tools for explicit computations of cohomology of functors:

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

**Theorem 1.1.** (*Kan extensions*) Let  $\mathcal{C}, \mathcal{D}$  be small categories and let  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  be a covariant functor. Restriction along  $\Phi$  induces an exact covariant functor

$$\Phi^* : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{D}}$$

mapping a covariant functor  $\mathcal{A} : \mathcal{C} \rightarrow \text{Mod}(k)$  to  $\mathcal{A} \circ \Phi$ . Moreover,  $\Phi^*$  has a left adjoint

$$\Phi_* : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{D}}$$

and a right adjoint

$$\Phi_! : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{D}}$$

By the usual abstract nonsense,  $\Phi^*$  preserves projectives and  $\Phi_!$  preserves injectives, but none of them is exact in general. Given a covariant functor  $\mathcal{A} : \mathcal{C} \rightarrow \text{Mod}(k)$  the functors  $\Phi_*(\mathcal{A}), \Phi_!(\mathcal{A}) : \mathcal{D} \rightarrow \text{Mod}(k)$  are called the *left* and *right Kan extension* of  $\mathcal{A}$ , respectively. The cohomology of functors on  $\mathcal{C}$  and  $\mathcal{D}$  is related by a Grothendieck spectral sequence (whose homology version is in [6]):

**Theorem 1.2.** Let  $\mathcal{C}, \mathcal{D}$  be small categories and let  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  be a covariant functor. For any functor  $\mathcal{A}$  in  $\hat{\mathcal{C}}$  there is a cohomology spectral sequence

$$E_2^{p,q} = H^p(\mathcal{D}; R^q \Phi_!(\mathcal{A})) \Rightarrow H^{p+q}(\mathcal{C}; \mathcal{A}) .$$

It is possible to compute the functors  $R^q \Phi_!(\mathcal{A})$  explicitly as follows. Let  $\mathcal{C}, \mathcal{D}$  be small categories and let  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  be a covariant functor. Given an object  $Y$  in  $\mathcal{D}$  we denote by  $\Phi^Y$  the category whose objects are pairs  $(X, \varphi)$  with  $X$  an object in  $\mathcal{C}$ ,  $\varphi : \Phi(X) \rightarrow Y$  in  $\mathcal{D}$ , and where a morphism  $(X, \varphi) \rightarrow (X', \varphi')$  in  $\Phi^Y$  is a morphism  $\alpha : X \rightarrow X'$  in  $\mathcal{C}$  such that  $\varphi' \circ \Phi(\alpha) = \varphi$ . We have a forgetful functor

$$\Phi^Y \longrightarrow \mathcal{C}$$

mapping  $(X, \varphi)$  to  $X$ ; for any  $\mathcal{F}$  in  $\mathbf{F}(\mathcal{C}; \mathcal{A})$  we denote by  $\mathcal{F}^Y$  the functor in  $\mathbf{F}(\Phi^Y; \mathcal{A})$  obtained from restricting  $\mathcal{F}$  through this forgetful functor. The homology version of the following Theorem can be found in [6]; the case  $q = 0$  is due to Kan.

**Theorem 1.3.** Let  $\mathcal{C}, \mathcal{D}$  be small categories, let  $\Phi : \mathcal{C} \rightarrow \mathcal{D}$  be a covariant functor and let  $k$  be a commutative ring. Let  $Y$  be an object in  $\mathcal{D}$ , let  $\mathcal{A}$  be a functor in  $\hat{\mathcal{C}}$ . For any integer  $q \geq 0$  we have a natural isomorphism

$$R^q \Phi_!(\mathcal{A})(Y) \cong H^q(\Phi^Y; \mathcal{A}^Y) .$$

In particular (for  $q = 0$ ) we have

$$\Phi_!(\mathcal{A})(Y) \cong \varinjlim_{\Phi^Y} (\mathcal{A}^Y) .$$

Proofs of 1.1, 1.2, 1.3 can, for instance, be found in [16, §2].

II. *EI*-CATEGORIES

**Definition 2.1.** (Lück [17]) An *EI*-category is a small category  $\mathcal{C}$  with the property that every endomorphism of any object in  $\mathcal{C}$  is an automorphism of that object.

For  $\mathcal{C}$  a category we denote by  $[\mathcal{C}]$  the set of isomorphism classes of objects in  $\mathcal{C}$ , and for any object  $X$  in  $\mathcal{C}$  we denote by  $[X]$  its isomorphism class in  $[\mathcal{C}]$ .

**Proposition 2.2.** *Let  $\mathcal{C}$  be an *EI*-category. Then the set  $[\mathcal{C}]$  is a partially ordered set with partial order defined by  $[X] \leq [Y]$  if  $\text{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ , for objects  $X, Y$  in  $\mathcal{C}$ .*

**Example 2.3.** Let  $G$  be a finite group,  $p$  a prime and let  $P$  be a Sylow- $p$ -subgroup of  $G$ . The *fusion system of  $G$  on  $P$*  is the category  $\mathcal{F} = \mathcal{F}_P(G)$  with objects the subgroups of  $P$  and group homomorphisms induced by conjugation in  $G$  as morphisms; that is,  $\text{Hom}_{\mathcal{F}}(Q, R) = \text{Hom}_G(Q, R) = \{\varphi : Q \rightarrow R \mid \text{there is } x \in G \text{ such that } \varphi(u) = xux^{-1} \text{ for all } u \in Q\}$  for any two subgroups  $Q, R$  of  $P$ . Clearly  $\mathcal{F}$  is an *EI*-category.

There is an obvious canonical functor  $\mathcal{C} \rightarrow [\mathcal{C}]$  sending an object  $X$  in  $\mathcal{C}$  to its isomorphism class  $[X]$ . The base change spectral sequence of this functor takes the following form (cf. [16, 1.1]):

**Theorem 2.4.** *Let  $\mathcal{C}$  be an *EI*-category and let  $k$  be a commutative ring. For any covariant functor  $\mathcal{A} : \mathcal{C} \rightarrow \text{Mod}(k)$  there is a cohomology spectral sequence*

$$E_2^{p,q} = H^p([\mathcal{C}]; \mathcal{A}_q) \Rightarrow H^{p+q}(\mathcal{C}; \mathcal{A}) ,$$

where, for  $q \geq 0$ , we denote by  $\mathcal{A}_q : [\mathcal{C}] \rightarrow \text{Mod}(k)$  the covariant functor sending  $[X]$  to  $H^q(\mathcal{C}_{\geq X}; \mathcal{A})$ .

III. SUBDIVISIONS OF *EI*-CATEGORIES

**Definition 3.1.** (cf. [14, 1.1]) The *subdivision of an *EI*-category  $\mathcal{C}$*  is the category  $S(\mathcal{C})$  defined as follows:

The objects of  $S(\mathcal{C})$  are the faithful functors  $\sigma : [m] \rightarrow \mathcal{C}$ , where  $m$  is a non-negative integer and  $[m] = \{0, 1, 2, \dots, m\}$  viewed as totally ordered set.

A morphism in  $S(\mathcal{C})$  from  $\sigma : [m] \rightarrow \mathcal{C}$  to  $\tau : [n] \rightarrow \mathcal{C}$  is a pair  $(\alpha, \mu)$  consisting of an injective monotone map  $\alpha : [m] \rightarrow [n]$  and an isomorphism of functors  $\mu : \sigma \cong \tau \circ \alpha$ .

Explicitly, an object of  $S(\mathcal{C})$  can be viewed as a chain of  $m$  composable non-isomorphism in  $\mathcal{C}$

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{n-1}} X_m$$

Any such chain is called a *chain of length  $m$* , where as usual chains of length 0 are just objects in  $\mathcal{C}$ .

A morphism in  $S(\mathcal{C})$  from such a chain of non-isomorphisms

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} X_m$$

to another chain of non-isomorphisms

$$\tau = Y_0 \xrightarrow{\psi_0} Y_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} Y_n$$

is a family  $\mu = (\mu_i)_{0 \leq i \leq m}$  where for each  $i$  there is  $\alpha(i) \in \{0, 1, \dots, n\}$  such that  $\mu_i : X_i \rightarrow Y_{\alpha(i)}$  is an isomorphism which makes the obvious diagrams commutative; that is,

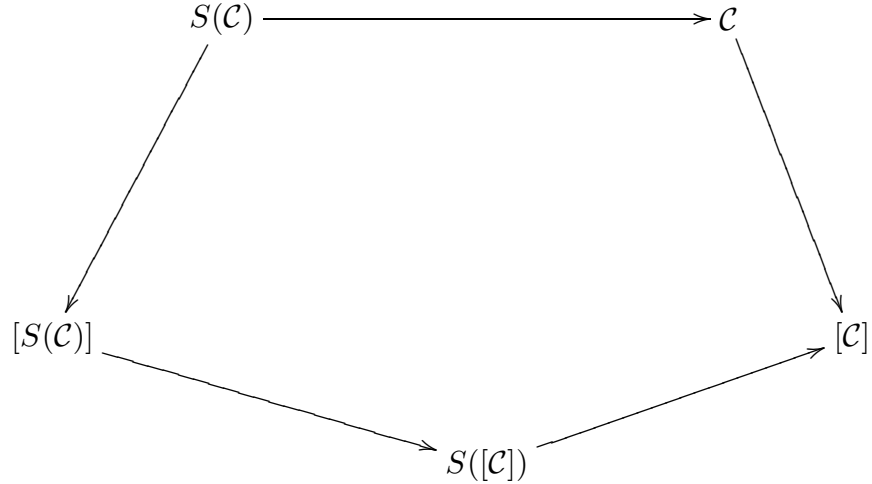
$$\mu_{i+1} \circ \varphi_i = \psi_{\alpha(i+1)-1} \circ \cdots \circ \psi_{\alpha(i)+1} \circ \psi_{\alpha(i)} \circ \mu_i$$

for any  $i \in \{0, 1, \dots, m-1\}$ . If  $\mathcal{C}$  is itself a poset then  $S(\mathcal{C})$  is the usual barycentric subdivision of  $\mathcal{C}$  as poset. The following is obvious:

**Proposition 3.2.** *Let  $\mathcal{C}$  be an EI-category. Then  $S(\mathcal{C})$  is an EI-category.*

In general, the five categories  $\mathcal{C}$ ,  $[\mathcal{C}]$ ,  $S(\mathcal{C})$ ,  $[S(\mathcal{C})]$  and  $S([\mathcal{C}])$  are related as follows:

**Proposition 3.3.** [14, 1.4] *Let  $\mathcal{C}$  be an EI-category. There is a commutative pentagon of canonical functors*



More precisely,

- the functor  $\mathcal{C} \rightarrow [\mathcal{C}]$  sends an object  $X$  in  $\mathcal{C}$  to its isomorphism class  $[X]$ ;
- The functor  $S(\mathcal{C}) \rightarrow [S(\mathcal{C})]$  sends a chain  $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m$  in  $S(\mathcal{C})$  to its isomorphism class  $[X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m]$

- the functor  $S(\mathcal{C})$  sends a chain  $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m$  in  $S(\mathcal{C})$  to its maximal term  $X_m$ ;
- the functor  $[S(\mathcal{C})] \rightarrow S([\mathcal{C}])$  sends  $[X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m]$  to  $[X_0] < [X_1] < \cdots < [X_m]$ ;
- the functor  $S([\mathcal{C}]) \rightarrow [\mathcal{C}]$  sends  $[X_0] < [X_1] < \cdots < [X_m]$  to  $[X_m]$ ;

**Remark 3.4.** Let  $\mathcal{C}$  be an *EI*-category.

(1)  $S([\mathcal{C}])$  is the barycentric subdivision of the poset  $[\mathcal{C}]$ ; in particular, the functor  $S([\mathcal{C}]) \rightarrow [\mathcal{C}]$  is a homotopy equivalence.

(2) It is easy to construct examples of *EI*-categories  $\mathcal{C}$  such that  $[\mathcal{C}]$  is contractible but  $[S(\mathcal{C})]$  is not.

**Definition 3.5.** Let  $\mathcal{C}$  be an *EI*-category. The partially ordered set  $[S(\mathcal{C})]$ , viewed as topological space, is called the *orbit space* of  $\mathcal{C}$ .

This definition is motivated by the fact that when  $\mathcal{C}$  is the fusion system of a finite group on one of its Sylow- $p$ -subgroups, then the resulting space  $[S(\mathcal{C})]$  coincides with the traditional notion of an orbit space; see Section V. Note that the spectral sequence 2.4 applies to both canonical functors  $\mathcal{C} \rightarrow [\mathcal{C}]$  and  $S(\mathcal{C}) \rightarrow [S(\mathcal{C})]$ . The following result from [14, 3.2] provides a tool to compute the cohomology of covariant functors on the orbit space:

**Theorem 3.6.** Let  $\mathcal{C}$  be an *EI*-category and let  $\mathcal{A} : [S(\mathcal{C})] \rightarrow \text{Mod}(k)$  be a covariant functor. There is a cochain complex  $C(\mathcal{A})$  of  $k$ -modules such that

$$C(\mathcal{A})^n = \bigoplus_{\substack{[\sigma] \in [S(\mathcal{C})] \\ |\sigma|=n}} \mathcal{A}([\sigma])$$

and such that

$$H^n(C(\mathcal{A})) \cong H^n([S(\mathcal{C})]; \mathcal{A})$$

for  $n \geq 0$ . In particular, if  $\mathcal{C}$  is finite then  $H^*([S(\mathcal{C})]; \mathcal{A})$  is bounded.

#### IV. FUSION SYSTEMS

**Definition 4.1.** (Puig, early 1990's) Let  $p$  be a prime and let  $P$  be a finite  $p$ -group. A *fusion system on  $P$*  is a category  $\mathcal{F}$  having as objects the subgroups of  $P$  and as morphisms sets of injective group homomorphisms  $\text{Hom}_{\mathcal{F}}(Q, R)$  between any two subgroups  $Q, R$  of  $P$ , with the following properties:

- composition of morphisms in  $\mathcal{F}$  is the usual composition of group homomorphisms;

- if  $\varphi : Q \rightarrow R$  is a morphism in  $\mathcal{F}$  then so is the induced isomorphism  $Q \cong \varphi(Q)$  and its inverse (which makes sense as  $\varphi$  is injective);
- $\text{Hom}_P(Q, R) \subseteq \text{Hom}_{\mathcal{F}}(Q, R)$ ; that is,  $\mathcal{F}$  contains all morphisms given by conjugation with elements in  $P$ ;
- (“Sylow-axiom”)  $\text{Aut}_P(P)$  is a Sylow- $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ ;
- (“Extension axiom”) if  $\varphi : Q \rightarrow P$  is a morphism in  $\mathcal{F}$  such that  $|N_P(\varphi(Q))| \geq |N_P(\varphi'(Q))|$  for any morphism  $\varphi' : Q \rightarrow P$  in  $\mathcal{F}$  then there is a commutative diagram of morphisms in  $\mathcal{F}$  of the form

$$\begin{array}{ccc} N_\varphi & \xrightarrow{\psi} & P \\ \iota \uparrow & & \parallel \\ Q & \xrightarrow[\varphi]{} & P \end{array}$$

where  $\iota$  is the inclusion morphism of  $Q$  into the group  $N_\varphi$  consisting of all  $y \in N_P(Q)$  for which there exists an element  $z \in N_P(\varphi(Q))$  satisfying  $\varphi(yuy^{-1}) = z\varphi(u)z^{-1}$  for all  $u \in Q$ .

Note that  $QC_P(Q) \subseteq N_\varphi \subseteq N_P(Q)$ ; the extension axiom implies that  $N_\varphi$  is the largest subgroup of  $N_P(Q)$  to which  $\varphi$  can be extended in the category  $\mathcal{F}$ . The following is an easy application of Sylow’s theorems:

**Proposition 4.2.** *Let  $G$  be a finite group, let  $p$  be a prime and let  $P$  be a Sylow- $p$ -subgroup of  $G$ . Then  $\mathcal{F}_P(G)$  is a fusion system.*

More generally, as a consequence of work of Alperin and Broué [1], every  $p$ -block of  $G$  gives rise to a fusion system. Not every fusion system occurs as that of a finite group (see work of Solomon [20], Levi-Oliver [12], Ruiz-Viruel [19]) and not even of a  $p$ -block (Kessar, Stancu [9], [10]). Except for the Solomon fusion system this requires the classification of finite simple groups.

**Question:** (D. J. Benson [3]) Can one associate with every fusion system  $\mathcal{F}$  a  $p$ -complete space which would coincide with  $BG_p^\wedge$  in case  $\mathcal{F} = \mathcal{F}_P(G)$  for some finite group  $G$ ?

The answer to that question is not known by the time of this writing, not even for fusion systems of  $p$ -blocks. Broto-Levi-Oliver developed in [4] the homotopy theory of fusion systems including a cohomological criterion for the existence and uniqueness of such a space.

## V. ORBIT SPACES OF FUSION SYSTEMS

Let  $G$  be a finite group, let  $p$  be a prime. Denote by  $\mathcal{P}$  the  $G$ -poset of non-trivial  $p$ -subgroups of  $G$  and by  $sd(\mathcal{P})$  its barycentric subdivision. That is,  $sd(\mathcal{P})$  consists of all chains  $Q_0 < Q_1 < \cdots < Q_m$  of non-trivial  $p$ -subgroups  $Q_i$ ,  $0 \leq i \leq m$ , ordered by inclusion of chains. This is obviously again a  $G$ -poset, and we denote by  $sd(\mathcal{P})/G$  the set of  $G$ -orbits in  $sd(\mathcal{P})$ . This is again a poset, with partial order induced by that in  $sd(\mathcal{P})$ . The following Theorem is due to P. Symonds [21], proving a conjecture of P. Webb [22].

**Theorem 5.1.** *With the above notation, the orbit space  $sd(\mathcal{P})/G$  is contractible when viewed as topological space.*

L. Barker [2] extended this to  $p$ -blocks of finite groups:

**Theorem 5.2.** *For any  $p$ -block  $b$  of a finite group  $G$  the  $G$ -orbit space  $sd(\mathcal{Q})/G$  of the  $G$ -poset  $\mathcal{Q}$  of non-trivial  $b$ -Brauer pairs is contractible.*

The orbit space  $sd(\mathcal{P})/G$  can be interpreted as the orbit space of an  $EI$ -category in the sense of Definition 3.5:

**Proposition 5.3.** *Let  $P$  be a Sylow- $p$ -subgroup of the finite group  $G$ , set  $\mathcal{F} = \mathcal{F}_P(G)$  and let  $\mathcal{C}$  be the full subcategory of  $\mathcal{F}$  consisting of all non-trivial subgroups of  $P$ . As before, let  $\mathcal{P}$  be the  $G$ -poset of non-trivial  $p$ -subgroups of  $G$ . There is a canonical isomorphism of partially ordered sets*

$$sd(\mathcal{P})/G \cong [S(\mathcal{C})] .$$

Theorem 5.1. and 5.2 can be generalised as follows (cf. [15, 1.1]):

**Theorem 5.4.** *Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $P$  and let  $\mathcal{C}$  be a full subcategory of  $\mathcal{F}$  closed under isomorphisms and supergroups. Then  $[S(\mathcal{C})]$  is contractible.*

In order to prove 5.4 one considers the subposets

$[S_{\triangleleft}(\mathcal{C})]$  of chains  $Q_0 < Q_1 < \cdots < Q_m$  of subgroups in  $\mathcal{C}$  such that  $Q_i$  is normal in  $Q_m$  for  $0 \leq i \leq m$ ,

$[S_{\Phi}(\mathcal{C})]$  of chains  $Q_0 < Q_1 < \cdots < Q_m$  of subgroups in  $\mathcal{C}$  such that  $Q_0$  contains the Frattini subgroup of  $Q_m$ .

One shows then that

- the inclusions  $[S_{\triangleleft}(\mathcal{C})] \subseteq [S(\mathcal{C})]$  and  $[S_{\Phi}(\mathcal{C})] \subseteq [S(\mathcal{C})]$  induce isomorphisms on cohomology;
- $[S_{\Phi}(\mathcal{C})]$  is acyclic (and hence so are  $[S_{\triangleleft}(\mathcal{C})]$  and  $[S(\mathcal{C})]$ );
- $[S_{\triangleleft}(\mathcal{C})]$  and  $[S(\mathcal{C})]$  are simply connected.

Curiously, the arguments in [15] do not show that  $[S_{\Phi}(\mathcal{C})]$  is simply connected. Still, one needs to consider  $[S_{\Phi}(\mathcal{C})]$  to show that  $[S(\mathcal{C})]$  is acyclic.

## VI. SOME NUMERICAL CONJECTURES IN MODULAR REPRESENTATION THEORY

Let  $G$  be a finite group and let  $k$  be an algebraically field of characteristic  $p > 0$ . We consider the following non-negative numbers:

$\mathbf{k}(G) = \dim_k(Z(kG))$  = number of ordinary irreducible characters of  $G$ ;

$\ell(G)$  = number of isomorphism classes of simple  $kG$ -modules

$\ell_0(G)$  = number of isomorphism classes of projective simple  $kG$ -modules.

**Alperin's weight conjecture** (AWC) states that for any finite group  $G$  we should have the equality

$$\ell(G) = \sum_Q \ell_0(N_G(Q)/Q)$$

with  $Q$  running over a set of representatives of the  $G$ -conjugacy classes of  $p$ -subgroups of  $G$  (including  $Q = 1$ , so that the summand  $\ell_0(G)$  appears on the right side)

The **Knörr-Robinson** reformulation of AWC in [11] states that AWC is equivalent to the equality, for any finite group  $G$ ,

$$\mathbf{k}(G) - \ell_0(G) = \sum_{[\sigma] \in sd(\mathcal{P})/G} (-1)^{|\sigma|} \mathbf{k}(N_G(\sigma)) .$$

Both Alperin's original formulation and the Knörr-Robinson version admit more precise formulations in terms of the block decomposition of the group algebra  $kG$ . The alternating sum suggests that there should be a complex behind all this - which gets us to one of the main motivations for developing the material in the previous sections: it is in fact possible to interpret this alternating sum as cohomology of a suitable functor on the orbit space.

There are more subtle numerical conjectures in block theory, due to Dade and Robinson, which we expect to have interpretations in functor cohomological terms over suitable orbit spaces of  $EI$ -categories derived from fusion systems.

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# A GENERALIZATION OF THE LYNDON–HOCHSCHILD–SERRE SPECTRAL SEQUENCE AND SOME OTHER SHORT STORIES

P. H. KROPHOLLER

## 1. COHOMOLOGICAL CONTINUITY

We shall take a lightening tour around the following definition concerning discrete groups.

**Definition 1.1.** Let  $G$  be a group and let  $n$  be an integer. We shall say that the cohomology functor  $H^n(G, \_)$  is *continuous* if and only if the natural map

$$\varinjlim H^n(G, M_\lambda) \rightarrow H^n(G, \varinjlim M_\lambda)$$

is an isomorphism for all choices of filtered colimit system  $(M_\lambda)$  of  $\mathbb{Z}G$ -modules.

The definition was put on the map by Bieri and Eckmann with the fundamental observation that connects it with the property  $\text{FP}_n$ . A group  $G$  is called  $\text{FP}_n$  if and only if there is a projective resolution  $P_* \twoheadrightarrow \mathbb{Z}$  over  $\mathbb{Z}G$  in which  $P_i$  is finitely generated as a  $\mathbb{Z}G$ -module for all integers  $i \leq n$ .

**Lemma 1.2.** Fix  $n \in \mathbb{N} \cup \{\infty\}$ . The group  $G$  is of type  $\text{FP}_n$  if and only if the functors  $H^i(G, \_)$  are continuous for all  $i < n$ .

One could of course make the same definition for many other functors. Typically it is useful to consider additive functors between abelian categories. So we could consider the same definition for homology instead of cohomology, but in this case the situation is very simple.

**Lemma 1.3.** The homology functors  $H_n(G, \_)$  are always continuous.

*Proof.* Think about singular homology of  $BG$ . Singular homology has compact supports. □

Here is a nice application of this fact to Poincaré duality groups, also discovered by Bieri and Eckmann.

**Definition 1.4.** The group  $G$  is called an  *$n$ -dimensional Poincaré duality group* ( $\text{PD}^n$ -group for short) if there is a  $\mathbb{Z}G$ -module  $\tilde{\mathbb{Z}}$  which is additively isomorphic to  $\mathbb{Z}$  and natural isomorphisms

$$H^i(G, M) \cong H_{n-i}(G, M \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}).$$

**Lemma 1.5.** Every  $\text{PD}^n$ -group is of type  $\text{FP}$ .

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*Proof.* The cohomology of a Poincaré duality group  $G$  inherits the continuity property from its homology. Thus  $G$  is of type  $\text{FP}_\infty$  by Lemma 1.2. Duality also shows that the cohomology of  $G$  vanishes in dimensions  $> n$  and therefore  $G$  has finite cohomological dimension. Taking the properties together we can deduce from Schanuel's Lemma that there is a finite projective resolution and we then say that  $G$  has type  $\text{FP}$ .  $\square$

There are many other insights like this which can be brought into view by using the Definition 1.1. We shall mention another very briefly at the end of this article.

## 2. GENERALIZING THE LHSSS

First, here is a more technical result in which continuity plays a rôle.

**Theorem 2.1.** *Let  $G$  be a finitely generated group of cohomological dimension  $n + 1$  which has a  $\text{PD}^n$ -subgroup  $H$ . Assume that*

( $\sharp$ )  *$H$  is commensurable with all of its conjugates.*

*Then*

- $G$  is of type  $\text{FP}$ ;
- for some subgroup  $H_0$  of finite index in  $H$ , the Schreier graph of the pair  $G, H_0$  has more than one end:

$$e(G, H_0) \geq 2;$$

- $G$  is the fundamental group of a finite graph of groups in which all vertex and all edge groups are commensurable with  $H$ .

We are going to discuss what this means and why some generalization of the spectral sequence will be desirable in proving it. In the next section we shall reveal the strategy for generalizing and using the Lyndon–Hochschild–Serre spectral sequence [LHSSS].

In our theorem, the assumption ( $\sharp$ ) should be regarded with suspicion. Two subgroups  $K$  and  $L$  of  $G$  are said to be *commensurable* if and only if their intersection  $K \cap L$  has finite index in both  $K$  and  $L$ . This explains what ( $\sharp$ ) means but it does not explain why we might be interested in the condition. This note is not going to be the place for any proper defence. It is natural to consider the set up with  $G$  and  $H$  regardless of whether or not ( $\sharp$ ) holds and our theorem concerns one case of this. Here we remark only that in the other cases there is a  $g \in G$  such that  $H \cap H^g$  has cohomological dimension  $\leq n - 1$  and then other arguments of geometric group theory can be brought to bear.

One of the interesting conclusions of the theorem concerns the Schreier graph. For this, choose any finite set  $S$  of generators of  $G$  and consider the Cayley graph  $\Gamma(G, X)$  which has vertex set  $G$  and in which we join an edge between two vertices  $g_1, g_2$  whenever their difference  $g_1 g_2^{-1}$  belongs to  $S \cup S^{-1}$ . There is an action of  $G$  on  $\Gamma$  and this restricts to an action of  $H$ . The Schreier graph of the pair is the quotient graph  $\Gamma/H$ . If there is a way to remove a finite number of edges from this graph so that we are left with at least two infinite components then we say that the graph has more than one end and we write  $e(G, H) \geq 2$ . Of course the definitions

appear to depend on the choice of generating set, but although that is true of the particular Schreier graph, the number of ends is nevertheless invariant. The theory of ends of graphs entered into geometric group theory in the late nineteen-sixties with the work of Stallings and Swan showing that groups of cohomological dimension one are free. The use of ends here is similar in spirit and is a key step to the third conclusion of the theorem. For the theory of ends to work effectively it is better when the group is finitely presented, or at the very least, of type  $FP_2$ . However, our  $G$  is only given to be finitely generated which is equivalent to type  $FP_1$ . Looking at the situation from what seems to be the wrong end of the telescope, we might also observe that the high dimensional cohomology functors are all continuous, indeed zero! It is a wonderful observation of Strebel that this is exactly the right way to look: we shall be able to prove that all the cohomology functors  $H^i(G, \ )$  are continuous by *downward induction on  $i$* .

So the conclusions of the theorem come in three steps. The first is necessary in order to prove the second and the second is necessary in order to prove the third. In this article we shall focus on how the first conclusion is proved.

Obviously  $(\#)$  holds if  $H$  is normal. In this special case we can also see an obvious possible approach to studying the cohomology of  $G$ , the LHSSS

$$E_2^{p,q} = H^p(G/H, H^q(H, M)) \Rightarrow H^{p+q}(G, M).$$

Condition  $(\#)$  is not so very far from normality.

### 3. GENERALIZING THE LHSSS

The clue to generalizing the LHSSS comes from viewing it as a Grothendieck spectral sequence. If  $K$  is a normal subgroup we have a factorization of the zeroth cohomology functor as a composite of two functors illustrated by a commutative triangle:

$$\begin{array}{ccc} \mathfrak{Mod}_{\mathbb{Z}G} & \xrightarrow{H^0(G, \ )} & \mathfrak{Ab} \\ & \searrow^{H^0(K, \ )} & \nearrow^{H^0(G/K, \ )} \\ & \mathfrak{Mod}_{\mathbb{Z}[G/K]} & \end{array}$$

The zeroth cohomology functor is simply the  $G$ -fixed point functor and the triangle expresses the fact that we can take  $G$ -fixed point in any object of  $\mathfrak{Mod}_{\mathbb{Z}G}$  (i.e. any  $\mathbb{Z}G$ -module) by first taking  $K$ -fixed point to obtain an object the category of modules for the quotient group  $G/K$  and then taking  $G/K$ -fixed points. All the categories here are abelian categories with enough injectives and the left hand functor carries injective modules to injective modules. We get a Grothendieck spectral sequence relating the derived functors and this is the classical LHSSS.

For our application we have a subgroup  $H$  which is not normal but is close to normal. There is no quotient group and so we need a new candidate to be the

intermediate category in our commutative triangle. The following definition is designed to solve this problem.

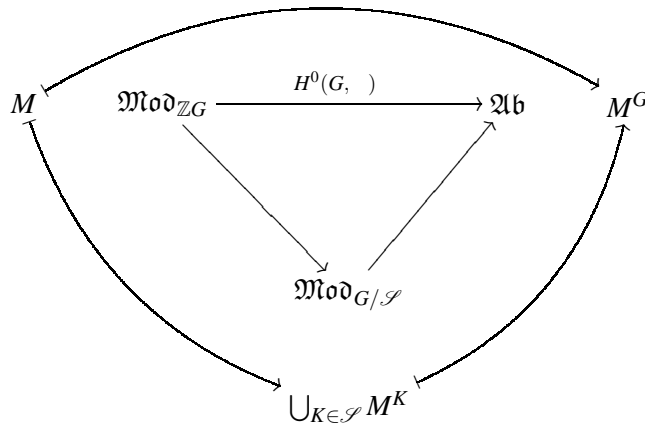
**Definition 3.1.** A set  $\mathcal{S}$  of subgroups of  $G$  is called *admissible* if and only if it satisfies the two conditions

- $\mathcal{S}$  is closed under conjugation;
- $\mathcal{S}$  is closed under finite intersections.

If  $\mathcal{S}$  is admissible then we define  $\mathfrak{Mod}_{G/\mathcal{S}}$  to be the full subcategory of  $\mathfrak{Mod}_{\mathbb{Z}G}$  comprising those objects  $M$  such that

$$M = \bigcup_{K \in \mathcal{S}} H^0(K, M).$$

Now we have a factorization of the fixed point functor!



Just like the other categories in this game, our new category is abelian and has enough injectives. The left hand functor does indeed carry injectives to injectives. We thus have a spectral sequence relating the right derived functors of the three “fixed point” functors in the diagram.

We can apply the spectral sequence to prove our theorem by taking  $\mathcal{S}$  to be the set of subgroups commensurable with  $H$ . Condition (#) guarantees that  $\mathcal{S}$  is an admissible set of subgroups.

The spectral sequence works very effectively if we choose projective modules for the coefficients. This is because projective modules are homologically acyclic and therefore in cohomology we find that  $H^i(H, P)$  is non-zero only if  $i = n$  for the  $PD^n$ -group  $H$ . In this way the spectral sequence collapses when we consider filtered colimit systems of projective modules. Since  $G$  has finite cohomological dimension we can now use downward induction on  $i$  to establish the continuity property with arbitrary filtered colimit systems of modules.

#### 4. CONCLUSIONS

In the lecture I also mentioned how the new spectral sequence can be used to establish the well known result  $H^*(F, \mathbb{Z}F) = 0$  where  $F$  is Thompson’s group.

Note that we do not need a set of commensurable subgroups to obtain a spectral sequence. The two conditions for admissibility allow for more general possibilities. There may well be other applications. For example we could consider the set of subgroups of  $SL_n(\mathbb{Q})$  which are commensurable with  $SL_n(\mathbb{Z})$  or we could consider the set of pointwise stabilizers of finite sets in the group of permutations of an infinite set.

It is natural to consider the limit

$$\lim_{\leftarrow} G/H$$

as  $H$  runs through an admissible set of subgroups. This inverse limit has a topology as well as inheriting a natural structure as a  $G$ -set. Interestingly there is always a natural monoid structure on this limit. The resulting monoid is not necessarily a group. For example, if  $G$  is the group of permutations of a countably infinite set  $X$  and  $\mathcal{S}$  is the set of pointwise stabilizers of finite sets then the limit is naturally isomorphic to the monoid of injective maps  $X \rightarrow X$ .

We have not fully explored the rôle of this limit object. It may well be that at least in the cases when it is a group then its continuous cohomology is closely related or even isomorphic to the derived functor cohomology suggested by the spectral sequence viewpoint.

For the details, references and proofs see [1].

We have seen that by continuity considerations lead to some interesting developments. Here is another instance. Take any group  $G$  and consider the set

$$\mathcal{C}(G) = \{n \geq 1 : H^n(G, \ ) \text{ is continuous}\}.$$

In general we know very little about this set. However, if we restrict to the class  $\mathfrak{H}\mathfrak{F}$  of hierarchically decomposable groups which I introduced some ten years ago then we have the following dichotomy.

**Theorem 4.1.** *For a group  $G$  in  $\mathfrak{H}\mathfrak{F}$ , the set  $\mathcal{C}(G)$  is either finite or cofinite in the set of positive integers.*

This tantalizing fact leads to some very interesting questions. Even for very elementary and natural classes of groups it seems very difficult to determine which groups belong to the *finite* and which to the *cofinite* side of this dichotomy.

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**AN EXTENSION OF THE CLASS OF COMPACT LIE GROUPS:  
ABSTRACT**

NITU KITCHLOO

In this note, we study a class of topological groups that extends the class of Lie groups. Most of the results stated below may be found in [1], which constitutes joint work with C. Broto. We show that groups in this class share many of the homological properties of Lie groups. Let us begin by describing this class of topological groups.

**Definition 0.1.** For a topological group  $H$ , the classifying space for proper actions,  $\underline{EH}$  is a  $H$ -CW complex with the property that all the isotropy subgroups are compact Lie group, and given a compact subgroup  $K \subseteq H$ , the fixed point space  $\underline{EH}^K$  is contractible.

Define a class  $\mathcal{C}$  consisting of topological groups  $H$  such that  $\underline{EH}$  may be constructed as a *finite  $H$ -CW complex*. Notice that if  $H$  is a compact Lie group, then we may take  $\underline{EH}$  to be a point, hence  $\mathcal{C}$  is indeed an extension of the class of compact Lie groups.

1. PROPERTIES OF GROUPS IN  $\mathcal{C}$

Here we state some theorems that can be shown to be true for groups in the class  $\mathcal{C}$ . Let  $K \in \mathcal{C}$ . We have

**Theorem 1.1.** For any prime  $p$ , the  $\mathbb{F}_p$ -algebra  $H^*(BK, \mathbb{F}_p)$  is finitely generated. Moreover, its Krull dimension is the rank of the largest elementary abelian  $p$ -group in  $K$ .

Given an elementary abelian  $p$ -group  $V$ , let  $T_V$  denote the Lannes  $T$ -functor on the category of unstable algebras over the mod- $p$  Steenrod algebra. We have:

**Theorem 1.2.** The following natural map is an isomorphism

$$T_V H^*(BK, \mathbb{F}_p) \longrightarrow \prod_{\varphi \in \text{Rep}(V, K)} H^*(BC_K(\varphi), \mathbb{F}_p)$$

where  $\text{Rep}(V, K)$  denotes the (finite) set of representations of  $V$  in  $K$ , and  $C_K(\varphi)$  is the centralizer in  $K$  of a representation  $\varphi$ .

The following theorem may be seen as a generalization of the above theorem:

**Theorem 1.3.** Given a finite  $p$ -group  $\pi$ , the following map is a homotopy equivalence:

$$BC_K(\pi)_p \longrightarrow \text{maps}(B\pi, (BK)_p^\wedge)$$

where  $X_p$  denotes the  $p$ -completion of a space  $X$ .

We only mention here that theorems like the centralizer decomposition theorem, and Quillen's  $\mathcal{F}$ -isomorphism theorem also hold for groups in the class  $\mathcal{C}$ .

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## 2. AN EXAMPLE

In this section, we would like to construct a nontrivial example of a group in the class  $\mathcal{C}$ . Let  $G$  be a connected, simply connected compact Lie group. Let  $LG$  denote the group of smooth loops in  $G$ .  $LG$  admits a universal central  $S^1$ -extension  $\mathbb{L}G$ . Consider the following theorem:

**Theorem 2.1.** *The topological affine Tits building*

$$\mathbf{A}(LG) := \text{hocolim}_I LG/H_I$$

of  $LG$  is  $\mathbb{T}\tilde{\times}LG$ -equivariantly contractible. In other words, given any compact subgroup  $K \subset \mathbb{T}\tilde{\times}LG$ , the fixed point space  $\mathbf{A}(LG)^K$  is contractible.

Here  $I$  runs over certain proper subsets of roots of  $G$ , and the  $H_I$  are certain compact ‘parabolic’ subgroups of  $LG$ . In particular, the indexing set is finite.

**Remark 2.2.** The natural action of the rotation group  $\mathbb{T}$  on  $LG$  lifts to  $\mathbb{L}G$ , and the  $\mathbb{T}$ -action preserves the subgroups  $H_I$ . Hence  $\mathbf{A}(LG)$  admits an action of  $\mathbb{T}\tilde{\times}LG$ , with the center acting trivially. We can therefore express  $\mathbf{A}(LG)$  as

$$\mathbf{A}(LG) = \text{hocolim}_I \mathbb{L}G/\mathbb{H}_I$$

where  $\mathbb{H}_I$  is the induced central extension of  $H_I$ .

OTHER DESCRIPTIONS OF  $\mathbf{A}(LG)$ 

This Tits building has other descriptions as well. For example:

1.  $\mathbf{A}(LG)$  can be seen as the classifying space for proper actions with respect to the class of compact Lie subgroups of  $\mathbb{T}\tilde{\times}LG$ .
2. It also admits a more differential-geometric description as the smooth infinite dimensional manifold of holonomies on  $S^1 \times G$ : Let  $\mathcal{S}$  denote the subset of the space of smooth maps from  $\mathbb{R}$  to  $G$  given by

$$\mathcal{S} = \{g(t) : \mathbb{R} \rightarrow G, g(0) = 1, g(t+1) = g(t) \cdot g(1)\};$$

then  $\mathcal{S}$  is homeomorphic to  $\mathbf{A}(LG)$ . The action of  $h(t) \in LG$  on  $g(t)$  is given by  $hg(t) = h(t) \cdot g(t) \cdot h(0)^{-1}$ , where we identify the circle with  $\mathbb{R}/\mathbb{Z}$ . The action of  $x \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$  is given by  $xg(t) = g(t+x) \cdot g(x)^{-1}$ .

3. The description given above shows that  $\mathbf{A}(LG)$  is equivalent to the affine space  $\mathcal{A}(S^1 \times G)$  of connections on the trivial  $G$ -bundle  $S^1 \times G$ . This identification associates to the function  $f(t) \in \mathcal{S}$ , the connection  $f'(t)f(t)^{-1}$ . Conversely, the connection  $\nabla_t$  on  $S^1 \times G$  defines the function  $f(t)$  given by transporting the element  $(0, 1) \in \mathbb{R} \times G$  to the point  $(t, f(t)) \in \mathbb{R} \times G$  using the connection  $\nabla_t$  pulled back to the trivial bundle  $\mathbb{R} \times G$ .

**Remark 2.3.** These equivalent descriptions have various useful consequences. For example, the model given by the space  $\mathcal{S}$  of holonomies says that given a finite cyclic group  $H \subset \mathbb{T}$ , the fixed point space  $\mathcal{S}^H$  is homeomorphic to  $\mathcal{S}$ . Moreover, this is a homeomorphism of  $LG$ -spaces, where we consider  $\mathcal{S}^H$  as an  $LG$ -space and identify  $LG$  with  $LG^H$  in the obvious way. Notice also that  $\mathcal{S}^{\mathbb{T}}$  is  $G$ -homeomorphic to the model of the adjoint representation of  $G$  defined by  $\text{Hom}(\mathbb{R}, G)$ .

Finally, the description of  $\mathbf{A}(LG)$  as the affine space  $\mathcal{A}(S^1 \times G)$  implies that the fixed point space  $\mathbf{A}(LG)^K$  is contractible for any compact subgroup  $K \subseteq \mathbb{T}\tilde{\times}LG$ .

*It follows from the above remark that*

**Theorem 2.4.** *The space  $\mathbf{A}(LG)$  is a model for  $\underline{EH}$ , where  $H = \mathbb{L}G$ . In particular, the groups  $\mathbb{L}G$  and  $LG$  belong to  $\mathcal{C}$ .*

*We are still limited by a lack of examples of groups in the class  $\mathcal{C}$ .*

### 3. QUESTIONS

*We end this note with a few questions.*

**Question.** Let  $H \subseteq G$  be a subgroup. Assume that this inclusion is a homotopy equivalence. Then does it follow that  $G$  is in  $\mathcal{C}$  if  $H$  is? If this is true, then Hatcher's result would show for example, that  $\text{Diff}(S^3)$  belongs to  $\mathcal{C}$ .

**Question.** What can we expect of groups in the class  $\mathcal{C}(\mathcal{C})$ . In other words, what properties can we expect of topological groups  $G$  that admit finite  $G$ -CW complexes with isotropy groups belonging to  $\mathcal{C}$ , and contractible fixed point spaces under any subgroup  $H$  which belongs to  $\mathcal{C}$ ?

**Question.** Construct nontrivial examples of groups in  $\mathcal{C}$ . The example of loop groups above is a special case of a family of examples known as Kac-Moody groups. An affirmative answer to the first question would also provide many new examples.

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# TRIANGULATED TRACK CATEGORIES

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ABSTRACT. This is a summary of my talk at the Skye Conference on Pure and Applied Topology 2005. The talk was a report on a joint work with Hans-Joachim Baues where we give a new insight to triangulated category theory from the viewpoint of groupoid-enriched categories and cohomology of categories.

In this note we define the most general notion introduced in [BM] of a 2-dimensional triangulated category (i. e. a triangulated track category). We also state the main theorem connecting this 2-categorical concept with classical triangulated category theory. Afterwards we present the main example which is the homotopy 2-category of a stable model category. The note intends to be self-contained, but some background on elementary 2-category theory is required. We refer the reader to [Bor]. At the end of the note we briefly sketch the connections with cohomology of categories.

A groupoid-enriched category or *track category*  $\mathbf{B}$  is a 2-category where all 2-morphisms are invertible with respect to vertical composition. The 2-morphisms in track categories are also termed *tracks*. Horizontal composition is denoted by juxtaposition, and we use the symbol  $\square$  for the vertical composition. The arrow  $\rightarrow$  stands for morphisms and (pseudo)functors, and  $\Rightarrow$  denotes tracks. Identity morphisms are designated by 1, and the symbol  $0^\square$  is used for identity tracks.

The automorphism group  $\text{Aut}_{\mathbf{B}}(f)$  of a morphism  $f$  in a track category  $\mathbf{B}$  is the set of tracks  $f \Rightarrow f$  with sum given by the vertical composition. A *strict zero object*  $*$  in a track category  $\mathbf{B}$  is an object such that the morphism groupoids  $\text{Hom}_{\mathbf{B}}(*, X)$ ,  $\text{Hom}_{\mathbf{B}}(X, *)$  are always the trivial groupoid, in particular zero morphisms  $0: X \rightarrow * \rightarrow Y$  in  $\mathbf{B}$  are defined.

Recall that a *pseudofunctor* between track categories  $\varphi: \mathbf{C} \rightarrow \mathbf{B}$  is an assignment of objects, maps and tracks which preserves composition and identities only up to certain given tracks

$$\varphi_{f,g}: \varphi(f)\varphi(g) \Rightarrow \varphi(fg) \text{ and } \varphi_X: \varphi(1_X) \Rightarrow 1_{\varphi(X)}.$$

These tracks must satisfy well-known coherence and naturality properties. If  $\mathbf{C}$  and  $\mathbf{B}$  have a strict zero object  $*$  we say that  $\varphi$  is *\*-normalized* if  $\varphi$  preserves the

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strict zero object  $\varphi(*) = *$  and  $\varphi_{0,f} = 0^\square$  and  $\varphi_{0,f} = 0^\square$  are identity tracks for all morphisms  $f$  in  $\mathbf{B}$ .

**Definition 1.** Let  $\mathbf{A}$  be an additive category and  $t: \mathbf{A} \xrightarrow{\sim} \mathbf{A}$  and additive equivalence. A *good translation track category*  $(\mathbf{B}, s)$  over  $(\mathbf{A}, t)$  consists of

- (1) a track category  $\mathbf{B}$  with a strict zero object  $*$ .
- (2) A functor  $p: \mathbf{B} \rightarrow \mathbf{A}$  from the underlying ordinary category of  $\mathbf{B}$  such that  $p$  is the identity on objects and  $p(f) = p(g)$  if and only if there exists a track  $f \Rightarrow g$ , i. e.  $\mathbf{A}$  is the *homotopy category* of  $\mathbf{B}$ .
- (3) Isomorphisms

$$\sigma_f: \text{Hom}_{\mathbf{A}}(tX, Y) \cong \text{Aut}_{\mathbf{B}}(f)$$

for all morphisms  $f: X \rightarrow Y$  in  $\mathbf{B}$  such that given a track  $\alpha: f \Rightarrow g$  in  $\mathbf{B}$ ,

$$\alpha \square \sigma_f(x) = \sigma_g(x) \square \alpha;$$

and given composable morphisms  $\bullet \xrightarrow{h} \bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$  in  $\mathbf{B}$

$$f \sigma_g(x) = \sigma_{fg}(p(f)x) \text{ and } \sigma_g(x)h = \sigma_{gh}(x(tp(h))).$$

- (4) A  $*$ -normalized pseudofunctor  $s: \mathbf{B} \rightarrow \mathbf{B}$  such that for all maps  $f: X \rightarrow Y$  in  $\mathbf{B}$  and  $x \in \text{Hom}_{\mathbf{A}}(tX, Y)$  the equalities  $p(s(f)) = t(p(f))$  and  $s(\sigma_f(x)) = \sigma_{s(f)}(-t(x))$  hold, i. e.  $s$  induces  $t$  in the homotopy category and  $-t$  in  $\text{Hom}_{\mathbf{A}}(t, -)$  through the isomorphisms  $\sigma$ .

**Definition 2.** A *track triangle* in a good translation track category  $(\mathbf{B}, s)$  is a diagram in  $\mathbf{B}$

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & \curvearrowright & \uparrow H_0 & \curvearrowright & \uparrow H_2 & \curvearrowright & \\ A & \xrightarrow{f} & B & \xrightarrow{i} & C & \xrightarrow{q} & sA & \xrightarrow{s(f)} & sB \\ & & \downarrow H_1 & & & & & & \\ & & 0 & & & & & & \end{array}$$

such that the following equalities hold

$$\begin{aligned} (qH_0) \square (H_1^\square f) &= \sigma_0(1_{tA}), \\ (s(f)H_1) \square (H_2^\square i) &= \sigma_0(-1_{tB}), \\ (s(i)H_2) \square ((s(H_0) \square s_{i,f})^\square q) &= \sigma_0(1_{tC}). \end{aligned}$$

A morphism of track triangles is a diagram in  $\mathbf{B}$

$$(3) \quad \begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{i} & C & \xrightarrow{q} & sA \\ \downarrow k_0 & \swarrow K_0 & \downarrow k_1 & \swarrow K_1 & \downarrow k_2 & \swarrow K_2 & \downarrow sk_0 \\ \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{i}} & \bar{C} & \xrightarrow{\bar{q}} & s\bar{A} \end{array}$$

such that the next three equalities are satisfied

$$\begin{aligned} k_2 H_0 &= (\bar{H}_0 k_0) \square (\bar{i} K_0) \square (K_1 f), \\ s(k_0) H_1 &= (\bar{H}_1 k_1) \square (\bar{q} K_1) \square (K_2 i), \\ s(k_1) H_2 &= (\bar{H}_2 k_2) \square (s(\bar{f}) K_2) \square ((s_{\bar{f}, k_0}^\square \square s(K_0) \square s_{k_1, f}) q). \end{aligned}$$

Here  $\bar{H}_i$  are the structure tracks of the lower track triangle in (3).

**Definition 4.** A *pretriangulated track category* is a good translation track category  $(\mathbf{B}, s)$  over  $(\mathbf{A}, t)$  together with a distinguished family of track triangles which is subject to the following axiom:

(TTr1) For each morphism  $f: A \rightarrow B$  in  $\mathbf{B}$  there exists a distinguished track triangle

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & \uparrow H_0 & \curvearrowright & \uparrow H_2 & \curvearrowright & \\
 A & \xrightarrow{f} & B & \xrightarrow{i} & C & \xrightarrow{q} & sA \xrightarrow{s(f)} sB \\
 & & \downarrow H_1 & & & & \\
 & & 0 & & & & 
 \end{array}$$

It is a *triangulated track category* if in addition the following axiom is satisfied:

(TTr2) Given distinguished track triangles

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & \uparrow H_0 & \curvearrowright & \uparrow H_2 & \curvearrowright & \\
 A & \xrightarrow{f} & B & \xrightarrow{i} & C & \xrightarrow{q} & sA \xrightarrow{s(f)} sB \\
 & & \downarrow H_1 & & & & \\
 & & 0 & & & & 
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & \uparrow \bar{H}_0 & \curvearrowright & \uparrow \bar{H}_2 & \curvearrowright & \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{i}} & \bar{C} & \xrightarrow{\bar{q}} & s\bar{A} \xrightarrow{s(\bar{f})} s\bar{B} \\
 & & \downarrow \bar{H}_1 & & & & \\
 & & 0 & & & & 
 \end{array}$$

any diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 k_0 \downarrow & \swarrow K_0 & \downarrow k_1 \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{B}
 \end{array}$$

in  $\mathbf{B}$  extends to a track triangle morphism

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{i} & C & \xrightarrow{q} & sA \\
 k_0 \downarrow & \swarrow K_0 & \downarrow k_1 & \swarrow K_1 & \downarrow k_2 & \swarrow K_2 & \downarrow sk_0 \\
 \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{i}} & \bar{C} & \xrightarrow{\bar{q}} & s\bar{A}
 \end{array}$$

**Theorem 5** ([BM] IV.2.17). *Let  $\mathbf{A}$  be an additive category,  $t: \mathbf{A} \xrightarrow{\sim} \mathbf{A}$  an additive equivalence and  $(\mathbf{B}, s)$  a good translation track category over  $(\mathbf{A}, t)$ . If  $(\mathbf{B}, s)$  admits a (pre)triangulated track category structure then the triangles  $\mathbf{A}$*

$$A \xrightarrow{p(f)} B \xrightarrow{p(i)} C \xrightarrow{p(q)} tA$$

induced by arbitrary track triangles

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & \uparrow H_0 & \curvearrowright & \uparrow H_2 & \curvearrowright & \\
 A & \xrightarrow{f} & B & \xrightarrow{i} & C & \xrightarrow{q} & sA & \xrightarrow{s(f)} & sB . \\
 & & \downarrow H_1 & & & & & & \\
 & & 0 & & & & & & 
 \end{array}$$

in  $(\mathbf{B}, s)$  are the exact triangles of a (pre)triangulated category structure on  $\mathbf{A}$  in the classical sense.

*Remark 6.* Notice that in the hypothesis of Theorem 5 only the *existence* of a (pre)triangulated track category structure in  $(\mathbf{B}, s)$  is required. It is not necessary to specify it. Moreover, the exact triangles in  $\mathbf{A}$  given by the theorem are induced by *arbitrary* track triangles in  $(\mathbf{B}, s)$ , not only by the distinguished ones of a certain (pre)triangulated track category structure on  $(\mathbf{B}, s)$ .

*Example 7.* Let  $\mathbf{C}$  be a pointed stable Quillen model category, or more generally cofibration category. The category  $\mathbf{C}$  has a zero object  $*$  and the suspension functor  $\Sigma$  in the homotopy category  $\mathrm{Ho} \mathbf{C}$  is an additive equivalence

$$\Sigma: \mathrm{Ho} \mathbf{C} \xrightarrow{\sim} \mathrm{Ho} \mathbf{C}.$$

The *homotopy track category*  $\mathrm{Ho}_2 \mathbf{C}$  of  $\mathbf{C}$  is defined as follows. Objects and morphisms are the same as in  $\mathbf{C}$ , and tracks  $H: f \Rightarrow g$  between maps  $f, g: A \rightarrow B$  with  $A$  cofibrant and  $B$  fibrant are homotopy classes of homotopies  $IA \rightarrow B$  from  $f$  to  $g$  relative to the boundary of the cylinder  $A \vee A \subset IA$ . The suspension induces a pseudofunctor

$$\Sigma_2: \mathrm{Ho}_2 \mathbf{C} \longrightarrow \mathrm{Ho}_2 \mathbf{C}.$$

Moreover,  $(\mathrm{Ho}_2 \mathbf{C}, \Sigma_2)$  is a good translation track category over  $(\mathrm{Ho} \mathbf{C}, \Sigma)$  and admits a triangulated track category structure. The distinguished track triangles are given by cofiber sequences. The structure tracks are described in the slides of the talk. The slides are available in the web page of the conference <http://www.abdn.ac.uk/~wpe006/conference/skye/present.php>.

In [BM] III.5 we define the *translation cohomology* of an additive category  $\mathbf{A}$  equipped with an additive equivalence  $t: \mathbf{A} \xrightarrow{\sim} \mathbf{A}$

$$H^*(\mathbf{A}, t).$$

A good translation track category  $(\mathbf{B}, s)$  over  $(\mathbf{A}, t)$  determines a 3-dimensional characteristic class in translation cohomology,

$$\langle \mathbf{B}, s \rangle \in H^3(\mathbf{A}, t).$$

Actually any cohomology class in  $H^3(\mathbf{A}, t)$  arises in this way, see [BM] III.6.3. The triangles induced in  $\mathbf{A}$  by track triangles in  $(\mathbf{B}, s)$  in the sense of Theorem 5 can be directly obtained from the characteristic class  $\langle \mathbf{B}, s \rangle$ , see [BM] V.5.7. Moreover, in [BM] V.7.1 we give purely cohomological conditions on  $\langle \mathbf{B}, s \rangle$  which are equivalent to the existence of a (pre)triangulated track category structure in  $(\mathbf{B}, s)$ . For this we use a spectral sequence for the computation of the translation cohomology of products constructed in [BM] V.8.

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# Lusternik-Schnirelmann category of Orbifolds

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## Abstract

The idea is to generalize to the case of orbifolds the classical Lusternik-Schnirelmann theory. This paper defines a notion of LS-category for orbifolds. We show that some of the classical estimates for the regular category have their analogue in the case of orbifolds. We examine the topic in some detail using a mixture of approaches from equivariant theory and foliations.

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## 1 Orbifolds

The concept of orbifold was introduced by Satake [17] in 1956 under the name of V-manifold and developed later by Thurston [18] and Haefliger [10]. In this section we review some basic definitions concerning orbifolds.

Let  $X$  be a Hausdorff space.

An *orbifold chart* for an open set  $V \subset X$  is a triple  $(\tilde{V}, G, \varphi)$  such that

1.  $\tilde{V}$  is a connected open subset of  $\mathbf{R}^n$ ,
2.  $G$  is a finite subgroup of diffeomorphisms of  $\tilde{V}$ ,
3.  $\varphi: \tilde{V} \rightarrow V$  is a  $G$ -invariant map inducing a homeomorphism from  $\tilde{V}/G$  onto  $V$ .

If  $V_i \subset V_j$ , an *injection*  $(\lambda_{ij}, h_{ij}): (\tilde{V}_i, G_i, \varphi_i) \rightarrow (\tilde{V}_j, G_j, \varphi_j)$  between the orbifold charts is:

1. an embedding  $\lambda_{ij}: \tilde{V}_i \rightarrow \tilde{V}_j$  such that  $\varphi_j \circ \lambda_{ij} = \varphi_i$  and
2. an injective homomorphism  $h_{ij}: G_i \rightarrow G_j$  with  $\lambda_{ij}$  equivariant respect to  $h_{ij}$ , i.e.  $\lambda_{ij}(gx) = h_{ij}(g)\lambda_{ij}(x)$  for all  $g \in G_i$ ,  $x \in \tilde{V}_i$ .

An *orbifold atlas* on  $X$  is a family  $\{(\tilde{V}_i, G_i, \varphi_i)\}_{i \in I}$  of orbifold charts such that

1.  $\mathcal{V} = \{V_i\}_{i \in I}$  is a covering of  $X$
2. If  $V_i \subset V_j$ , then there exists an injection  $(\lambda_{ij}, h_{ij}): (\tilde{V}_i, G_i, \varphi_i) \rightarrow (\tilde{V}_j, G_j, \varphi_j)$
3. For any point  $x \in V_i \cap V_j$ , with  $V_i, V_j \in \mathcal{V}$ , there exists  $W \in \mathcal{V}$  such that  $x \in W \subset V_i \cap V_j$ .

The open sets  $V_i \in \mathcal{V}$  are called *basic* open sets.

Two atlases are *equivalent* if they are contained in the same maximal atlas.

An *orbifold*  $X$  of dimension  $n$  is a Hausdorff space equipped with an equivalent class of orbifold atlases.

A manifold  $M$  is a special case of orbifold where every group  $G_i$  is the trivial group  $G_i = \{1\}$  for all  $i \in I$ . This orbifold structure on  $M$  will be called the *trivial orbifold structure*.

A typical example of orbifold is provided by a properly discontinuous action of a group  $G$  on a manifold  $\tilde{M}$ . The quotient space  $\tilde{M}/G$  has an orbifold structure inherited from the manifold structure of  $\tilde{M}$ . The orbifold charts are obtained by suitable restrictions of the quotient map  $\tilde{M} \rightarrow \tilde{M}/G$ . In this case,  $\tilde{M}/G$  is called a *global* orbifold.

To each point  $x$  in an orbifold  $X$  we can associate a finite subgroup  $G_x$  of  $GL_n(\mathbf{R})$ , well defined up to conjugacy. Let  $V$  be an open basic neighborhood of  $x$  and consider the corresponding chart  $(\tilde{V}, G, \varphi)$ . Let  $\tilde{x}$  be any preimage of  $x$ ,  $\varphi(\tilde{x}) = x$ . The isotropy group  $G_{\tilde{x}}$  of  $\tilde{x}$  for the action of  $G$  on  $\tilde{V}$  is called the *local group* of  $x$  and denoted  $G_x$ . If  $W$  is another basic neighborhood of  $x$  and  $(\tilde{W}, H, \psi)$  with  $\psi(\tilde{w}) = x$  then the isotropy group  $H_{\tilde{w}}$  of  $\tilde{w}$  for the action of  $H$  on  $\tilde{W}$  is conjugate of  $G_x$ . Consider  $V \subset W$ , then there exists an injection  $(\lambda, h): (\tilde{V}, G, \varphi) \rightarrow (\tilde{W}, H, \psi)$  and the following diagram commutes:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\lambda} & \tilde{W} \\ \varphi \searrow & & \swarrow \psi \\ & V \subset W & \end{array}$$

We can identify  $G$  and its image by  $h$  since  $h$  is a monomorphism,  $h(G) = G$ . If  $g \in G_{\tilde{x}}$  we will show that there is  $\gamma \in H$  such that  $\gamma h(g) \gamma^{-1} \in H_{\tilde{w}}$ . Let  $g \in G_{\tilde{x}}$  and  $\lambda(\tilde{x}) = \tilde{y}$ . Then  $g\tilde{x} = \tilde{x}$  and there exists  $\gamma \in H$  such that  $\tilde{w} = \gamma\tilde{y}$ . We have:

$$h(g)\tilde{y} = h(g)\lambda(\tilde{x}) = \lambda(g\tilde{x}) = \lambda(\tilde{x}) = \tilde{y} \text{ and } \gamma h(g)\gamma^{-1}\tilde{w} = \gamma h(g)\gamma^{-1}\gamma\tilde{y} = \gamma h(g)\tilde{y} = \gamma\tilde{y} = \tilde{w}$$

Then the local group  $G_x$  of  $x$  is well defined up to conjugacy. We can think of  $G_x$  as the conjugacy class  $(G_{\tilde{x}})$  of  $G_{\tilde{x}}$  with  $\varphi(\tilde{x}) = x$ .

We will say that a point  $x \in X$  is *regular* if  $G_x$  is trivial, and *singular* otherwise. The set  $\Sigma X = \{x \in X \mid G_x \neq \{1\}\}$  is the *singular locus* of  $X$ . The singular locus is not an orbifold in general.

**PROPOSITION 1.1** [18] *The singular locus  $\Sigma X$  of an orbifold  $X$  is a closed set with empty interior.*

Observe that the set of regular points is a dense connected open subset of  $X$ .

**DEFINITION 1.2** *We call a reduced orbifold if  $G_x$  acts effectively for all  $x \in X$ .*

Every reduced orbifold is a quotient of a smooth manifold by an effective almost free action of a Lie group  $G$  [1]. If the group is discrete and the action is proper (for instance, if  $G$  is finite) the orbifold is a global quotient.

## 1.1 Orbifold maps

There are many definitions of orbifold maps. For our purposes we choose to use essentially the original definition of Satake. For each point in the orbifold, this definition implies the existence of an homomorphism between the isotropy groups which will play a key role in our definition of category.

A map  $f: X \rightarrow X'$  between orbifolds is a smooth *orbifold map* if for any point  $x \in X$  there is a basic open set  $V \subset X$  around  $x$  and a basic open set  $V' \subset X'$  around  $f(x)$  with corresponding charts  $(\tilde{V}, G, \varphi)$  and  $(\tilde{V}', G', \varphi')$  verifying the following conditions:

1. there exists a smooth map  $\tilde{f}: \tilde{V} \rightarrow \tilde{V}'$  and
2. an homomorphism  $\bar{f}: G \rightarrow G'$

such that  $\varphi' \tilde{f} = f \varphi$  and  $\tilde{f}$  is equivariant with respect to  $\bar{f}$ :

$$\tilde{f}(gx) = \bar{f}(g)\tilde{f}(x) \text{ for all } g \in G, x \in \tilde{V}.$$

An open set  $U \subset X$  is an orbifold with the orbifold structure induced by the orbifold structure on  $X$ .

The inclusion map  $i: U \rightarrow X$  is an orbifold map with  $\tilde{f} = \text{id}: \tilde{V} \rightarrow \tilde{V}$  and  $\bar{f} = \text{id}: G \rightarrow G$ . The constant map  $c: X \rightarrow X'$  with  $c(x) = x_0 \in X' \forall x \in X$  is an orbifold map considering  $\tilde{f}$  and  $\bar{f}$  to be constant maps, that is  $\tilde{f}: \tilde{V} \rightarrow \tilde{V}'$  such that  $\tilde{f}(\tilde{x}) = \tilde{x}_0$  for all  $\tilde{x} \in \tilde{V}$  where  $\varphi(\tilde{x}) = x$  and  $\varphi'(\tilde{x}_0) = x_0$  and  $\bar{f}: G \rightarrow G'$  such that  $\bar{f}(g) = 1$  for all  $g \in G$ .

For any two orbifolds  $X$  and  $X'$ , the cartesian product  $X \times X'$  is an orbifold with the orbifold structure given by the product of the orbifolds structures of  $X$  and  $X'$ . This structure on  $X \times X'$  is called the *product orbifold structure*.

A homotopy  $H: X \times I \rightarrow Y$  between orbifolds is an *orbifold homotopy* if  $H$  is a smooth *orbifold map* considering  $I$  with the trivial orbifold structure and  $X \times I$  with the product orbifold structure.

## 2 Categorical sets for orbifolds

We describe an open subset  $U \subset X$  as *orbifold categorical* if there is an orbifold homotopy  $H: U \times I \rightarrow X$  such that  $H_0: U \rightarrow X$  is inclusion and  $H_1: U \rightarrow X$  is the constant map. Here  $U$  is regarded as an orbifold with the orbifold structure induced by the one on  $X$ . In other words, the open subset  $U$  of  $X$  is orbifold categorical if the inclusion  $U \hookrightarrow X$  factors through a point up to orbifold homotopy.

**DEFINITION 2.1** *The orbifold category  $\text{cat}_{\text{orb}}X$  of an orbifold  $X$  is the least number of orbifold categorical open sets required to cover  $X$ . If no such covering exists we say that the orbifold category is infinite.*

Note that in general, the orbifold category  $\text{cat}_{\text{orb}}X$  of an orbifold  $X$  does not coincide with the ordinary category  $\text{cat}X$  of the underlying topological space. In general,  $\text{cat}X \leq \text{cat}_{\text{orb}}X$ .

The orbifold category is an invariant of orbifold homotopy type that coincides with the ordinary category when the orbifold structure is trivial.

If  $U$  is an orbifold categorical open set, let  $H:U \times I \rightarrow X$  be the orbifold homotopy. For any point  $(x,t) \in U \times I$ , let  $y = H(x,t)$ . There is a basic open set  $V_x \subset U$  around  $x$ , an open set  $I_t \in I$  around  $t$  and a basic open set  $V_y \subset X$  around  $y$  with  $G_x$  and  $G_y$  the corresponding isotropy groups. The homotopy  $H$  is locally lifted to a homotopy  $\tilde{H}_{(x,t)}:\tilde{V}_x \times I_t \rightarrow \tilde{V}_y$  which is equivariant with respect to the homomorphism  $\bar{H}_{(x,t)}:G_x \rightarrow G_y$ .

**PROPOSITION 2.2** *Let  $H:U \times I \rightarrow X$  be an orbifold homotopy of the inclusion. Then  $\bar{H}_{(x,t)}:G_x \rightarrow G_y$  is injective for all  $x \in U$  and  $t \in I$  with  $y = H(x,t)$ .*

**Proof:** See [3]. The proof is based in the known fact that every orbifold is isomorphic to the space of leaves of a foliation with compact leaves and finite holonomy groups [15].  $\square$

**COROLLARY 2.3** *Let  $H:U \times I \rightarrow X$  be an orbifold homotopy of the inclusion. If  $y = H(x,t)$ , then  $G_x$  is a subgroup of  $G_y$ . That is, an orbifold homotopy preserves the isotropy.*

**Proof:** Since  $\bar{H}_{(x,t)}:G_x \rightarrow G_y$  is injective, we have that  $G_x = \text{im}\bar{H}_{(x,t)}$  which is a subgroup of  $G_y$ .  $\square$

### 3 Stratification

Let  $\mathcal{S}_H = \{x \in X \mid G_x = H\}$ . For each  $x \in X$ , let  $\mathcal{S}_x$  be the connected component of  $\mathcal{S}_{G_x}$  containing  $x$ . We define a stratification  $\mathcal{S}$  of  $X$  as  $\mathcal{S} = \{\mathcal{S}_x\}_{x \in X}$ . That is, the strata of  $\mathcal{S}$  are the connected components of the sets with the same isotropy. Since the regular set is connected, we have that for all the regular points  $x \in X - \Sigma X$  the stratum  $\mathcal{S}_x = \mathcal{S}_1 = X - \Sigma X$  coincides with the regular set.

**PROPOSITION 3.1** [19] *Let  $G$  be a compact Lie group acting on a compact manifold  $M$ . Then*

1. *the action has finitely many conjugacy classes of isotropy groups.*
2.  *$\mathcal{S}_{(H)} = \{z \in M \mid (G_z) = (H)\}$  is a submanifold of  $M$  which may have components of different dimension*

We can always consider all the local isotropy groups as subgroups of a Lie group  $G$ . Now we introduce a filtration  $\mathcal{H}$  of the set  $\mathcal{H}_0$  of all the isotropy groups of the action of  $G$ :

$$\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_r = \{G\}$$

where  $\mathcal{H}_{i+1} = \{H \leq G \mid \exists H_i \in \mathcal{H}_i \text{ such that } H \not\supseteq H_i\}$  for each  $0 \leq i \leq r-1$ .

Now, we introduce an *orbifold filtration* of  $X$ :

$$X = X^0 \supset X^1 \supset \cdots \supset X^r$$

where

$$X^{i+1} = \{x \in X^i \mid G_x \in \mathcal{H}_{i+1}\}$$

Alternately, we can see each term of the filtration as  $X^i = \bigcup_{H \in \mathcal{H}_i} \mathcal{S}_H$ .

From Proposition 3.1 follows that the length of the filtration is finite. Note that each  $X^i$  is a union of orbifolds and may have several connected components.

We will prove now that an orbifold homotopy of the inclusion preserves the orbifold filtration.

**PROPOSITION 3.2** *Let  $U$  be an open set of  $X$  and  $H:U \times I \rightarrow X$  be an orbifold homotopy of the inclusion. If  $\{X^i \mid i = 1 \dots r\}$  is the orbifold filtration, then  $H_t(X^i \cap U) \subset X^i$  for all  $t \in I$ .*

**Proof:** If  $x \in X^i$  then  $G_x \in \mathcal{H}_i$  since  $x$  is in the connected component of  $\mathcal{S}_{G_x}$  containing  $x$ . Therefore, there is a subgroup  $H \in \mathcal{H}_{i-1}$  such that  $G_x \not\supseteq H$ . By Corollary 2.3,  $G_x$  is a subgroup of  $G_{H(x,t)}$  then  $H \not\supseteq G_x \subset G_{H(x,t)}$  with  $H \in \mathcal{H}_{i-1}$  and  $H(x,t) \in X^i$ .  $\square$

Now we give a lower bound for the orbifold category in terms of the category of each term of the filtration.

**PROPOSITION 3.3**  $\text{cat}_{\text{orb}} X^i \leq \text{cat}_{\text{orb}} X$  for  $i = 1, \dots, r$

**Proof:** Let  $U$  be an orbifold categorical open set for  $X$  and  $H:U \times I \rightarrow X$  be an orbifold homotopy of the inclusion with  $H_1(U) = x_0$ . By Proposition 3.2 we have that the restriction of  $H$  to  $X^i$  has image in the same term of the filtration:

$$H: (X^i \cap U) \times I \rightarrow X^i$$

then  $X^i \cap U$  is an orbifold categorical open set for  $X^i$  and  $\text{cat}_{\text{orb}} X^i \leq \text{cat}_{\text{orb}} X$ .

Even if  $X^i$  is not an orbifold, we can extend the definition of  $\text{cat}_{\text{orb}} X^i$  in the obvious way: the sum of the orbifold categories of its connected components.  $\square$

Then we have that the orbifold category is bounded below by the ordinary category of  $X$  and by the orbifold category of the exceptional set:

$$\max\{\text{cat}X, \text{cat}_{\text{orb}}\Sigma X\} \leq \text{cat}_{\text{orb}}X$$

In the ordinary Lusternik-Schnirelmann theory we have that the dimension of a connected topological space is an upper bound for its category [13]:

$$\text{cat}X \leq \dim X + 1$$

This upper bound is not true in general for the orbifold category since we can produce examples of orbifolds of a fixed dimension and arbitrary large orbifold category. For instance, consider  $G = \mathbf{Z}_2$  acting by rotation on a genus  $g$  surface. The quotient orbifold is topologically a sphere and the singular set consists of  $2g + 2$  points. Therefore  $\text{cat}_{\text{orb}}\Sigma X = 2g + 2$  and  $\text{cat}_{\text{orb}}X \geq 2g + 2$ . We can construct a categorical covering by  $2g + 2$  open sets, then  $\text{cat}_{\text{orb}}X = 2g + 2$  while the dimension of the orbifold is 2.

We give the following version of the dimensional upper bound for the orbifold category:

**PROPOSITION 3.4** *If  $\Sigma X \neq \emptyset$ , then  $\text{cat}_{\text{orb}}X \leq \text{cat}_{\text{orb}}\Sigma X + \dim X$ .*

**Proof:** Let  $\{V_1, \dots, V_m\}$  be an orbifold categorical covering of  $\Sigma X$ . For each  $V_i \subset \Sigma X$ , there exists an open subset  $U_i$  of  $X$  such that  $V_i \subset U_i$  and  $U_i$  is orbifold categorical [5]. Then  $\{U_1, \dots, U_m\}$  is an orbifold categorical covering of  $\Sigma X$  by open sets of  $X$ .

If  $n = \dim X$ , the regular set of the orbifold,  $X - \Sigma X$ , is an open manifold of dimension  $n$ . We will show that  $X - \Sigma X$  is deformable into a  $(n - 1)$ -dimensional simplicial complex and then,  $\text{cat}_{\text{orb}}(X - \Sigma X) \leq n$ .

Take a small triangulation of  $X$  such that the intersection of  $\Sigma X$  with each top dimensional simplex is either contractible or empty [2, 11]. For each top dimensional simplex with empty intersection choose a point  $p$  in its interior, let  $P$  be the set of these points. Choose a path  $\gamma_p$  in  $X$  for each  $p \in P$ , which never crosses itself, joining this point  $p$  with a point in  $\Sigma X$  such that  $\gamma_p(I) \cap \gamma_q(I) \subset \Sigma X$  for all  $q \in P$ ,  $q \neq p$ . Let

$$\Gamma = \bigcup_{p \in P} \gamma_p(I)$$

So,  $X - \Sigma X$  has the same type of homotopy than  $X - \Sigma X \setminus \Gamma$ .

The open subset  $X - \Sigma X \setminus \Gamma$  is deformable into its  $(n - 1)$ -simplicial structure  $T$ , so its ordinary category is bounded by the ordinary category of  $T$ . Therefore, by the classical dimensional bound for the ordinary category [12], we have  $\text{cat}(X - \Sigma X \setminus \Gamma) \leq n$ . Then  $\text{cat}(X - \Sigma X) \leq n$ . Since  $X - \Sigma X$  is a manifold, we have that  $\text{cat}(X - \Sigma X) = \text{cat}_{\text{orb}}(X - \Sigma X)$  and we can choose  $\{U_{m+1}, \dots, U_{m+n}\}$  an orbifold categorical covering of  $X - \Sigma X$  and  $\{U_1, \dots, U_m, U_{m+1}, \dots, U_{m+n}\}$  is an orbifold categorical covering of  $X$ .  $\square$

The following example shows that these estimates are optimal.

**EXAMPLE 3.5** Consider the action of  $G = \mathbf{Z}_3$  on the projective plane  $\mathbf{RP}^2$  given by the rotation by  $2\pi/3$  on the covering 2-sphere  $\mathbf{S}^2$ .

The rotation has 2 fixed points on  $\mathbf{S}^2$ , denoted by  $\{\pm a\}$ . As  $G$  has odd order, the quotient action on  $\mathbf{RP}^2$  has a unique fixed-point, denoted by  $[a]$ . Then  $\Sigma X = [a]$  and  $\text{cat}_{\text{orb}}\Sigma X = 1$ . The orbifold  $X$  is topologically  $\mathbf{RP}^2$  and the ordinary category of  $X$  is  $\text{cat}X = 3$ . Thus,

$$3 = \text{cat}X \leq \text{cat}_{\text{orb}}X \leq \text{cat}_{\text{orb}}\Sigma X + \dim X = 1 + 2 = 3$$

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# MOST RANK TWO FINITE GROUPS ACT FREELY ON A HOMOTOPY PRODUCT OF TWO SPHERES

MICHAEL A. JACKSON

Recall that the  $p$ -rank of a finite group  $G$ ,  $\text{rk}_p(G)$ , is the largest rank of an elementary abelian  $p$ -subgroup of  $G$  and that the rank of a finite group  $G$ ,  $\text{rk}(G)$ , is the maximum of  $\text{rk}_p(G)$  taken over all primes  $p$ . We define the homotopy rank of a finite group  $G$ ,  $h(G)$ , to be the minimal integer  $k$  such that  $G$  acts freely on a finite CW-complex  $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$ . Benson and Carlson [3] have conjectured that for any finite group  $G$ ,  $\text{rk}(G) = h(G)$ . The case of their conjecture, when  $G$  is a rank one group, is a direct result of Swan's theorem [9]. Benson and Carlson's conjecture has also been verified by Adem and Smith [1] for rank two  $p$ -groups as well as for all rank two finite simple groups except  $PSL_3(\mathbb{F}_p)$  for  $p$  an odd prime.

Here we verify Benson and Carlson's conjecture for most finite groups of rank two. A result of Heller [5] states that if  $h(G) = 2$ , then  $\text{rk}(G) = 2$ . It follows that the conjecture holds for a given rank two group  $G$  if  $G$  acts freely on a finite CW-complex  $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ . We will find such actions using a recent result of Adem and Smith.

**Theorem 1** (Adem and Smith [1]). *Let  $G$  be a finite group and let  $X$  be a finitely dominated, simply connected  $G$ -CW complex such that every nontrivial isotropy subgroup has rank one. Then for some large integer  $N$  there exists a finite CW-complex  $Y \simeq \mathbb{S}^N \times X$  with a free action of  $G$  on  $Y$  such that the projection  $Y \rightarrow X$  is  $G$ -equivariant.*

Theorem 1 states that if a group  $G$  acts on a finite CW-complex  $X \simeq \mathbb{S}^m$ , such that the nontrivial isotropy subgroups are rank one, then  $h(G) \leq 2$ . Adem and Smith have also simplified the process of finding such actions, but we need to include two definitions before stating their result.

**Definition 2.** *Let  $\varphi : BG \rightarrow BU(n)$  be a map and let  $\alpha \in H^{2n}(BU(n), \mathbb{Z})$  be the top Chern class (Euler class) of  $U(n)$ . The Euler class in  $H^{2n}(BG, \mathbb{Z})$  associated to  $\varphi$  is  $\varphi^*(\alpha)$ .*

**Definition 3.** *A cohomology class  $\alpha \in H^*(BG, \mathbb{Z})$  is called effective if for elementary abelian subgroup  $E \subseteq G$  with  $\text{rk}(E) = \text{rk}(G)$ ,  $\text{res}_E^G(\alpha) \neq 0$ .*

**Theorem 4** (Adem and Smith [1]). *Let  $G$  be a finite group with  $\text{rk}(G) = m$ . If the Euler class associated to some map  $\varphi : BG \rightarrow BU(n)$  is effective, then  $G$  acts on a finite CW-complex  $X \simeq \mathbb{S}^{2n-1}$  such that the isotropy groups have rank at most  $m - 1$ .*

Applying Theorem 4 to rank two groups leads to Corollary 5.

**Corollary 5.** *Let  $G$  be a finite group with  $\text{rk}(G) = 2$ . If the Euler class associated to some map  $\varphi : BG \rightarrow BU(n)$  is effective, then  $h(G) = 2$ .*

In light of Corollary 5, verifying Benson and Carlson's conjecture for a rank two group  $G$  can be reduced to finding a particular map  $\varphi : BG \rightarrow BU(n)$  with an effective Euler class. Two

properties of maps from  $BG$  to  $BU(n)$  follow:

If  $\varphi$  and  $\mu$  are homotopic maps from  $BG$  to  $BU(n)$ , then the Euler classes corresponding to  $\varphi$  and  $\mu$  are the same. Dwyer and Zabrodsky [4] also have shown that for any  $p$ -group  $P$ ,  $\text{Rep}(P, U(n)) \xrightarrow{\cong} [BP, BU(n)]$  where  $\text{Rep}(P, U(n)) = \text{Hom}(P, U(n))/\text{Inn}(U(n))$ , and the map is  $\rho \mapsto B\rho$ ; therefore, if  $\varphi$  is a map  $BG \rightarrow BU(n)$  and if  $P$  is a  $p$ -subgroup of  $G$  for some prime  $p$ , then  $\varphi|_{BP}$  is induced by a unitary representation of  $P$ .

In light of Dwyer and Zabrodsky's result, we will be relating maps from  $BP$  to  $BU(n)$  to representations and then to characters. To do so, we must introduce the following notation:

$G_p$  will denote a Sylow  $p$ -subgroup of  $G$ ;  $\text{Char}_n(G_p)$  will be the set of degree  $n$  complex characters of  $G_p$ ; and  $\text{Char}_n^G(G_p)$  will be the subset of  $\text{Char}_n(G_p)$  consisting of those degree  $n$  complex characters of  $G_p$  that are the restrictions of class functions on  $G$ , meaning that they respect fusion in  $G$ .

We now define a map  $\psi_G : [BG, BU(n)] \rightarrow \prod_{p||G} \text{Char}_n^G(G_p)$ , by using the following diagram:

$$\begin{array}{ccccc} [BG, BU(n)] & \xrightarrow{\cong} & \prod_{p||G} [BG, BU(n)_p^\wedge] & \xrightarrow{\text{res}} & \prod_{p||G} [BG_p, BU(n)_p^\wedge] \\ \bar{\psi}_G \downarrow & & \bar{\phi}_G \downarrow & & \cong \uparrow \\ \prod_{p||G} \text{Char}_n(G_p) & \xlongequal{\quad} & \prod_{p||G} \text{Char}_n(G_p) & \xleftarrow{\cong} & \prod_{p||G} \text{Rep}(G_p, U(n)). \end{array}$$

Notice that spaces in the center and at the right of the top row contain  $BU(n)_p^\wedge$ , which is the  $p$ -completion of the space  $BU(n)$ . The bijection in the upper left is a result by Jackowski, McClure, and Oliver [6] while the bijection on the far right follows from the previously mentioned work of Dwyer and Zabrodsky [4]; the bijection in the lower right is a basic result in representation theory. The restriction map  $\text{res}$  is induced by the inclusion of the Sylow  $p$ -subgroups  $G_p$  into  $G$ . We now let maps  $\bar{\psi}_G$  and  $\bar{\phi}_G$  be the maps that make the diagram commute. The images of  $\bar{\psi}_G$  and  $\bar{\phi}_G$  both lie in the subset  $\prod_{p||G} \text{Char}_n^G(G_p) \subseteq \prod_{p||G} \text{Char}_n(G_p)$ ;

therefore, we let  $\psi_G$  and  $\phi_G$  be the maps  $\bar{\psi}_G$  and  $\bar{\phi}_G$ , respectively, with the range restricted to  $\prod_{p||G} \text{Char}_n^G(G_p)$ . We get the following result concerning the map  $\psi_G$ :

**Theorem 6** (Jackson [7, Theorem 1.3]). *If  $G$  is a finite group of rank two, then the natural mapping  $\psi_G : [BG, BU(n)] \rightarrow \prod_{p||G} \text{Char}_n^G(G_p)$  is a surjection.*

Using Theorem 6, we see that a map from  $BG$  to  $BU(n)$  with an effective Euler class can be demonstrated by giving appropriate characters in  $\text{Char}_n^G(G_p)$  for each prime  $p$  dividing the order of  $G$ , which leads to Definition 7.

**Definition 7.** Let  $G$  be a finite group,  $p$  a prime dividing  $|G|$ , and  $G_p$  a Sylow  $p$ -subgroup of  $G$ . A character  $\chi$  of  $G_p$  is called a  $p$ -effective character of  $G$  if  $\chi \in \text{Char}_n^G(G_p)$  and if for each elementary abelian subgroup  $E \subseteq G_p$  with  $\text{rk}(E) = \text{rk}(G)$ , the trivial character of  $E$  is not an irreducible summand of the character  $\chi|_E$ .

Theorem 8 relates  $p$ -effective characters and effective Euler classes.

**Theorem 8** (Jackson[8]). Let  $G$  be a finite group. If for each prime  $p$  dividing  $|G|$  there exists a  $p$ -effective character of  $G$ , then there is a map  $\varphi : BG \rightarrow BU(n)$  whose associated Euler class is effective.

**Corollary 9.** Let  $G$  be a finite group with  $\text{rk}(G) = 2$ . If for each prime  $p$  dividing the order of  $G$  there exists a  $p$ -effective character of  $G$ , then  $h(G) = 2$ .

Corollary 9 has reduced the process of showing that a rank two group has homotopy rank two to finding  $p$ -effective characters for each prime  $p$  dividing the order of the group. A definition from group theory is necessary in demonstrating the existence of  $p$ -effective characters.

**Definition 10.** Let  $G$  be a finite group, and  $H$  and  $K$  subgroups such that  $H \subset K$ . We say that  $H$  is strongly closed in  $K$  with respect to  $G$  if for each  $g \in G$ ,  $H^g \cap K \subseteq H$ .

We are now able to show a sufficient condition for the existence of a  $p$ -effective character.

**Proposition 11.** Let  $G$  be a finite group,  $p$  a prime divisor of  $|G|$ , and  $G_p$  a Sylow  $p$ -subgroup of  $G$ . If  $H \subseteq Z(G_p)$  exists such that  $H$  is non-trivial and strongly closed in  $G_p$  with respect to  $G$ , then  $G$  has a  $p$ -effective character.

Next we define a group that does not have a  $p$ -effective character for an odd prime  $p$ . We see that this group is the lone obstruction to a rank two group having a  $p$ -effective character. Let  $T_p \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_{\theta} \text{SL}_2(\mathbb{F}_p)$  where the action  $\theta$  is given by the obvious inclusion  $\text{SL}_2(\mathbb{F}_p) \rightarrow \text{GL}_2(\mathbb{F}_p) \cong \text{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p)$ . Notice that this group has been known as  $Qd(p)$  by group theorists. Also notice that for  $p$  an odd prime,  $T_p \subseteq \text{PSL}_3(\mathbb{F}_p)$ .

We now can show that most rank two finite groups have a  $p$ -effective character for any odd prime  $p$ .

**Theorem 12** (Jackson [8]). Let  $G$  be a finite group,  $p > 2$  a prime with  $\text{rk}_p(G) = \text{rk}(G) = 2$ , and  $G_p \in \text{Syl}_p(G)$ . If  $\Omega_1(Z(G_p))$  is not strongly closed in  $G_p$  with respect to  $G$ , then  $G$  has a subgroup  $H$  such that  $H/O_{p'}(H) \cong T_p$ .

Now we will turn our attention to the prime two. We begin with a restatement of a theorem of Alperin, Brauer, and Gorenstein [2, Proposition 7.1].

**Theorem 13** (Alperin, Brauer, and Gorenstein [2]). Let  $G$  be a finite group such that  $\text{rk}_2(G) = 2$  and let  $G_2 \in \text{Syl}_2(G)$ . If  $\Omega_1(Z(G_2))$  is not strongly closed in  $G_2$  with respect to  $G$ , then  $G_2$  is either dihedral, semi-dihedral, or wreathed.

As a result of Theorem 13, a rank two finite group has a 2-effective character if its Sylow 2-subgroups are not dihedral, semi-dihedral or wreathed. The cases of dihedral, semi-dihedral and wreathed Sylow 2-subgroups are shown to have 2-effective characters in Theorem 14.

**Theorem 14** (Jackson [8]). *If  $G$  is a finite group with a dihedral, semi-dihedral, or wreathed Sylow 2-subgroup such that  $\text{rk}(G) = 2$ , then  $G$  has a 2-effective character.*

To prove the dihedral and semi-dihedral case we need to demonstrate that there exists a character of the 2-group  $P$  that is 2-effective for any group containing  $P$  as a Sylow 2-subgroup. The demonstrated characters are constant on elements of the same order and so respect fusion in any group containing  $P$  as a Sylow 2-subgroup. In the case of a wreathed Sylow 2-subgroup, there are four cases that need to be worked through separately.

The last two theorems together yield Corollary 15.

**Corollary 15.** *For any rank two finite group  $G$ ,  $G$  has a 2-effective character.*

Corollary 15 and Theorem 12 lead us to our main result, Theorem 16.

**Theorem 16** (Jackson). *Let  $G$  be a finite group such that  $\text{rk}(G) = 2$ .  $G$  acts freely on a finite CW-complex  $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$  unless for some odd prime  $p$ ,  $G$  contains a subgroup  $H$  such that  $H/O_p(H) \cong T_p$ . In particular, if  $G$  is a finite group of rank two that does not contain a subgroup  $H$  such that  $H/O_p(H) \cong T_p$  for any odd prime  $p$ , then  $h(G) = 2$ .*

We conclude with comments on the groups  $T_p$ . It has been shown by Smith and Grodal as well as Unlu [10] that  $T_p$  for an odd prime  $p$  cannot act on a homotopy sphere with rank one isotropy groups. So the question of whether such groups have homotopy rank two cannot be approached using the methods developed by Adem and Smith. At this point it is unknown if  $T_p$  for an odd prime  $p$  has homotopy rank two.

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# CUBICAL COMPLEXES IN CONCURRENCY THEORY. DISCRETE AND CONTINUOUS MODELS FOR DIRECTED LIFTING PROBLEMS.

LISBETH FAJSTRUP

## 1. INTRODUCTION

In recent years, methods from algebraic topology and geometry have entered computer science. These methods are used in many different areas of computer science, and various traditional methods of geometry and algebraic topology are applied directly or customized to fit the applications.

Among the computer science disciplines which attract geometric methods is concurrency. Modern computers have more than one processor, and hence the execution of a program will often be distributed to different processors who then have to exchange information, to share memory, printers etc. For a fast execution, it is preferable that many processors work concurrently. On the other hand, the non-determinism introduced by various processors with their own local time and pace is problematic in verification that executions will do what they are expected to. Another problem introduced is the vast number of states in such a concurrent program - to check that a program will always behave correctly involves checking that all possible states are acceptable. This problem is referred to as the state space explosion problem.

To discuss the problems, one needs a good model for concurrency. A model describing a program with just one process is a labelled directed graph with loops and branchings - an automaton. In 1991, V. Pratt [9] introduced a Higher Dimensional Automaton, an HDA, the higher dimensional analogue of an automaton as a model for concurrency. An HDA is a cubical set, where a solid cube of dimension  $n$  represents  $n$  actions which may be executed concurrently. Fig. 1 shows two different representations of the execution of actions  $a$  and  $b$ . The first figure, with only boundary edges of the square, represents a choice



FIGURE 1. Two HDA, both modelling execution of the actions  $a$  and  $b$ . The first requires a sequential execution, whereas the second allows concurrent execution.

between “ $a$  then  $b$ ” or “ $b$  then  $a$ ”. The second figure with the full 2-cube has the interpretation that  $a$  and  $b$  may be executed concurrently by two processors. In particular,  $a$  and  $b$  do not interfere with each other and the result of the execution does not depend on the order in which  $a$  and  $b$  are executed.

## 2. HDA AND EXECUTIONS AS $\square$ -SETS AND DIRECTED PATHS.

**Definition 2.1.** An HDA is a cubical set, (a  $\square$ -set),  $M$ , i.e., a family of sets  $\{M_n | n \geq 0\}$  with face maps  $\partial_i^k : M_n \rightarrow M_{n-1}$  ( $1 \leq i \leq n$ ,  $k = -, +$ ) satisfying the semi-cubical relations:

$$\partial_i^k \partial_j^l = \partial_{j-1}^l \partial_i^k \quad (i < j)$$

A geometric cubical complex is a  $\square$ -set  $M$  such that for any pair  $L_n$  and  $K_m$  of elements of  $M$ , there is a (perhaps empty) common face  $F_r$  such that any other common face  $X_k$  is a face of  $F_r$ .

An HDA has an initial point  $p_0 \in M_0$ . A combinatorial execution of an HDA is a sequence of edges  $e_0, e_1, \dots, e_n \in M_1$ , where  $\partial_1^-(e_0) = p_0$  and  $\partial_1^-(e_i) = \partial_1^+(e_{i-1})$  for  $i = 1, \dots, n$ .

Concatenation of edges is denoted  $\star$ , so we will write  $e_0 \star e_1 \star \dots \star e_n$

Sometimes we also require an execution to end in a prescribed set of final points - the accepting states for the HDA.

**Definition 2.2.** The combinatorial equivalence relation on combinatorial execution paths is the closure (under concatenation and transitivity) of the relation  $\partial_1^- x \star \partial_2^+ x \sim \partial_2^- x \star \partial_1^+ x$  for all  $x \in M_2$ .

The geometric realisation of a geometric  $\square$ -set has the properties of a *locally partially ordered space*, an *lpo-space*, defined in [7]. Each cube inherits a partial order from  $\mathbb{R}^n$ . Take the transitive closure of these in all stars  $Star(v)$ , where  $v \in M_0$ . This yields a partial order in a neighborhood of any point, and they match on intersections in a more or less obvious way.

*Remark 2.3.* Notice that a simplex does not have an a priori partial order, and hence cubical sets are preferred from a directed topology point of view. Not every cubical subdivision of a space has a consistent local partial order, but after at most one subdivision, it has [4]. Moreover, a triangulation of a space may be cubically subdivided, so all triangulizable spaces can be equipped with an lpo-structure, which locally is as above.

A path in the geometric realization of a  $\square$ -set will only represent an execution, if it is a *dipath*,  $\gamma : \vec{I} \rightarrow \|M\|$ , i.e., continuous, locally increasing and initiating in  $p_0$ . A priori, this gives far too many execution paths - usually uncountably many, but up to equivalence, the geometric description agrees with the combinatorial one. We define (with notation as above)

**Definition 2.4.** Let  $\gamma_1$  and  $\gamma_2$  be dipaths in  $\|M\|$  and suppose that  $\gamma_1(0) = \gamma_2(0) \in M_0$ ,  $\gamma_1(1) = \gamma_2(1) \in M_0$ . Then  $\gamma_1$  is dihomotopic to  $\gamma_2$  if they are connected by a continuous family of dipaths with fixed initial and final point  $H : I \times \vec{I} \rightarrow \|M\|$ , locally increasing along the path parameter,  $t \in \vec{I}$ , but not along the homotopy parameter,  $s \in I$ .

$\vec{\pi}_1(X, x, y)$  is then defined as the set of equivalence classes of dipaths from  $x$  to  $y$  - notice that we need two base points. Because of the required direction, this gives rise not to fundamental groups but to a fundamental category. This category has been studied using different techniques [6], [8], [10],- adjusting the algebraic topology machinery to the new directed setting.

*Remark 2.5.* In general, for the study of topological spaces with a local direction is not possible to apply the well known tools. Guided by the needs in possible applications, one may try to enrich and modify classical tools and invariants so that they fit into the directed framework.

- Lemma 2.6.** (1) Suppose  $\gamma$  is a dipath in  $\|M\|$ , that  $\gamma(0), \gamma(1) \in M_0$ . Then there is a combinatorial dipath  $e_0, \dots, e_n$  whose geometric realization is dihomotopic to  $\gamma$ . If  $L_0, \dots, L_m$  is the sequence of carriers traversed by  $\gamma$  and  $v_-(x)$  is the minimal vertex of the cube corresponding to  $x$ , then the combinatorial dipath traverses  $v_-(L_0), \dots, v_-(L_m)$ .
- (2) Suppose the geometric realization of two combinatorial dipaths  $\gamma_1$  and  $\gamma_2$  are dihomotopic. Then they are also combinatorially equivalent.

**Proof.** See [5]. □

Hence up to equivalence, we have as many combinatorial dipaths as dipaths between vertices: Denote combinatorial dihomotopy classes  $\vec{\pi}_1^c(X, v, w)$ , then  $\vec{\pi}_1^c(\|M\|, v, w) = \vec{\pi}_1(\|M\|, v, w)$ , when  $v, w \in M_0$ .

### 3. DICOVERING SPACES - COMBINATORIAL AND CONTINUOUS

Two HDA are equivalent, if they support the “same” executions. In [11], this equivalence - bisimulation - is defined for HDA and compared to other models of concurrency. HDA are shown to be the most “expressive” models up to equivalence. In his thesis U. Fahrenberg has given a topological interpretation of the bisimulation relations in terms of a lifting property for dipaths and cubical dipaths. For details, see Fahrenbergs thesis [1].

The executions of an HDA, the dipaths in a  $\square$ -set, really live in the *universal discovering space*, and one would expect that bisimulation should be an equivalence of discovering spaces. We give a combinatorial version of a universal discovering space, which translates to the topological discovering space defined in [3] via geometric realization.

**Definition 3.1.** The universal discovering  $\square$ -set of a  $\square$ -set  $M = \{M_0, \dots, M_N\}, \partial_i^k$  with respect to a vertex  $v_0 \in M_0$  is given by  $(\tilde{M}, \tilde{\partial}_i^k)$  with  $\tilde{M} = \{\tilde{M}_0, \tilde{M}_1, \dots, \tilde{M}_N\}$  where  $\tilde{M}_n = \{([\gamma], x_n)\}, \gamma$  a combinatorial dipath from  $v_0$  to  $v_-(x_n), x_n \in M_n$  and  $[\gamma]$  is the combinatorial dihomotopy class of  $\gamma$ .

Boundary maps are given as:

$$\tilde{\partial}_k^+([\gamma], x_n) = ([\gamma \star e_k], \partial_k^+(x_n))$$

with  $e_k$  the edge in  $x_n$  from  $v_-(x_n)$  to  $v_-(\partial_k^+(x_n))$  (geometrically from  $(0, \dots, 0)$  to  $(0, \dots, 0, 1, 0, \dots, 0)$  with 1 in the  $k$ 'th coordinate) and

$$\tilde{\partial}_k^-([\gamma], x_n) = ([\gamma], \partial_k^-(x_n)).$$

In [3], we defined the universal discovering space of an lpo-space.

**Definition 3.2.** The universal discovering space of a local po-space  $X$  wrt.  $x_0 \in X$  is

$$\tilde{X}_{x_0} = \{[\gamma], \text{where } \gamma(0) = x_0\},$$

- dihomotopy classes with fixed endpoints, For a partially ordered  $U \subset X$  as in 3.3, and dipaths  $\gamma_i$  with  $\gamma_i(0) = x_0$  and  $\gamma_i(1) \in U$ , we say  $[\gamma_1] \sim_U [\gamma_2]$  if there is a dihomotopy  $H : I \times \vec{I} \rightarrow X$  s.t  $H_s(0) = x_0, H_s(1) \in U, H_0(t) = \gamma_1(t)$  and  $H_1(t) = \gamma_2(t)$ . Let  $U_{[\gamma]} = \{[\mu] \mid [\mu] \sim_U [\gamma]\}$ . For a covering  $\mathcal{U}$  as in 3.3, these sets define a topology on  $\tilde{X}_{x_0}$ .

In order to avoid (too) strange behaviour of  $\tilde{X}_{x_0}$ , we need some additional requirements:

**Definition 3.3.** An lpo-space  $X$  is locally relatively disconnected wrt.  $x_0 \in X$  if there is a basis for the topology  $\mathcal{U} = \{(U_i, \leq_i)\}, U_i \in X, (U_i, \leq_i)$  po-spaces s.t.

- $\mathcal{U}$  defines an lpo-structure on  $X$
- $x, y \in U_i$  implies  $|\vec{\pi}_1(U_i, x, y)| \leq 1$  (This ensures that locally  $p : \tilde{X}_{x_0} \rightarrow X$  is a continuous bijection of future sets.)
- $x \leq_i y$  if and only if  $|\vec{\pi}_1(U_i, x, y)| = 1$
- For all  $x \in X$  there is a  $U_i$  s.t. for  $[\gamma], [\eta] \in \vec{\pi}_1(X, x_0, x), [\gamma] = [\eta]$  if and only if  $[\gamma] \sim_{U_i} [\eta]$ . (This property ensures that  $\tilde{X}_{x_0}$  Hausdorff and that  $p$  has discrete fibers.)

$\tilde{X}_{x_0}$  is an lpo-space ( $[\gamma] \leq_{U_{[\mu]}} [\gamma \star \eta]$  if  $\eta : \vec{I} \rightarrow U$  and concatenation makes sense.) In general, there may still be directed loops in  $\tilde{X}_{x_0}$  - for peculiar examples, see [3].

*Remark 3.4.* The fibers of  $p$  are discrete, but their cardinalities may vary. In Fig.2, the first example is a two-cube with a hole in the middle, and its universal discovering; the fiber dimension increases from 1 to 2 as one enters the upper corner. The second example is the surface of the 3-cube with one of the

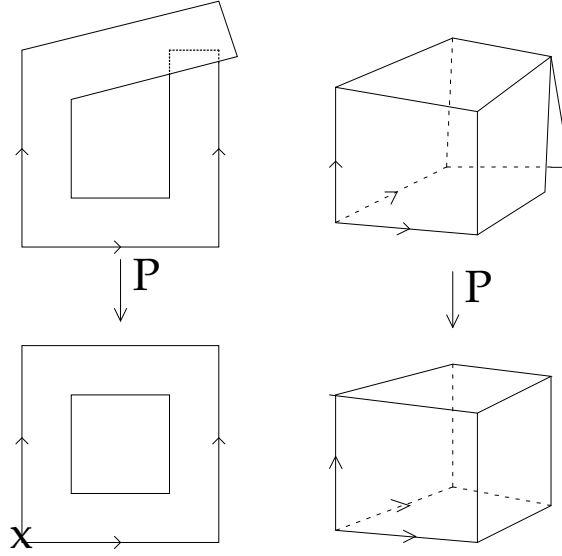


FIGURE 2. The fiberdimension is not constant in a dicovering.

lower 2-faces removed. In that case, the fiber dimension on the back edge is 2, but drops to 1 at the top vertex.

The geometric realization of  $\tilde{M}$  gives rise to an lpo-space  $\|\tilde{M}\|$ , which we want to compare with the lpo-space  $\|\widehat{M}\|_{v_0}$ . To define a relating map, recall that  $\|\tilde{M}\| = \coprod ([\gamma], x_i) \times \square_i / \sim$ , where the disjoint union is over all  $([\gamma], x_i) \in \tilde{M}_i$  and all  $i$ ,  $\square_i$  is the standard  $i$ -cube and the equivalence is defined via the boundary maps as usual.

**Definition 3.5.** Let  $([\gamma], x_n, w) \in ([\gamma], x_n) \times \square_n$ , then  $\phi([\gamma], x_n, w) = [\gamma \star l_w]$  where  $l_w$  is the line from  $\mathbf{0}$  to  $w$  in  $\square_n$ . This defines a map  $\phi : \|\tilde{M}\| \rightarrow \|\widehat{M}\|_{v_0}$

A converse map  $\psi : \|\widehat{M}\|_{v_0} \rightarrow \|\tilde{M}\|$  is defined as follows: Let  $\eta$  be a dipath in  $\|\widehat{M}\|$  from  $v_0$  and let  $\text{carrier}(\eta(1)) = x_k$ . Then  $\psi([\eta]) = (\alpha_*^{-1}(\eta), x_k, \eta(1))$ , where

- by an abuse of notation,  $\eta(1)$  will also denote the coordinates of the point  $\eta(1) \in \|\widehat{M}\|$  in the carrier  $\square_k$
- $\alpha_* : \vec{\pi}_1^c(v_0, v_-(x_k)) \rightarrow \vec{\pi}_1^c(v_0, \eta(1))$  is the isomorphism ( see [5] and [2] ) induced by concatenation with the line  $\alpha$  from  $v_-(x_k)$  to  $\eta(1)$ .

**Theorem 3.6.** For a geometric  $\square$ -set  $M$ , the map  $\phi : \|\tilde{M}\| \rightarrow \|\widehat{M}\|_{v_0}$  defined above induces a dihomoemorphism with inverse  $\psi$ , and in fact  $\phi : \text{Star}([\gamma], x_k) \rightarrow \text{Star}(x_k)_{[\gamma]}$  is a dihomoemorphism for all  $([\gamma], x_k) \in \tilde{M}_k$  and all  $k$ .

Sketch of proof: One has to see that stars of elements in  $M$  and “diconvex” subsets of these form a basis for the topology of  $\|\widehat{M}\|$  satisfying 3.3. The last condition in 3.3 is the harder one to prove. The others are straightforward.

In the definition of the inverse,  $\psi$ , we need the inverse  $\alpha_*^{-1}$ . This is constructed by taking a combinatorial approximation of the dipath  $\gamma \star l_{v_+}$  as in Lem. 2.6, where  $l_{v_+}$  is a line from  $\gamma(1)$  to the top vertex  $v_+(x_k)$  of its carrier. Finally eliminate the last piece of the combinatorial dipath, which, by construction, runs from  $v_-(x_k)$  to  $v_+(x_k)$ .

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# BIMODULES OVER OPERADS

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(JOINT WORK WITH PAUL-EUGÈNE PARENT AND JONATHAN SCOTT)

ABSTRACT. Let  $\mathcal{P}$  be any operad. A  $\mathcal{P}$ -bimodule  $\mathcal{R}$  that is a  $\mathcal{P}$ -cooperad induces a natural “fattening” of the category of  $\mathcal{P}$ -(co)algebras, expanding the morphism sets while leaving the objects fixed. The morphisms in the resulting  $\mathcal{R}$ -governed category of  $\mathcal{P}$ -(co)algebras can be viewed as morphisms “up to  $\mathcal{R}$ -homotopy” of  $\mathcal{P}$ -(co)algebras.

Let  $\mathcal{A}$  denote the associative operad in the category of chain complexes. We define a “diffracting” functor  $\Phi$  that produces  $\mathcal{A}$ -cooperads from symmetric sequences of chain coalgebras, leading to a multitude of “fattened” categories of (co)associative chain (co)algebras. In particular, we obtain a purely operadic description of the categories **DASH** and **DCSH** first defined by Gugenheim and Munkholm, via an  $\mathcal{A}$ -cooperad  $\mathcal{F}$  that is a minimal, free  $\mathcal{A}$ -bimodule resolution of  $\mathcal{A}$ . We show furthermore that the usual bar and cobar constructions are merely “shadows” cast by the diffraction functor.

Working with modules over operads further enables us to reformulate the duality of  $\mathcal{P}$ -algebras and  $\mathcal{P}$ -coalgebras as chirality of left and right  $\mathcal{P}$ -modules, eliminating the need for finite-type conditions.

This is a summary of the article [6], which can be found on the arXiv.

Over the past 30 years, operads have proven to be an excellent tool for encoding the multi-layered structure of certain classes of algebraic *objects*, e.g., the coherent systems of higher homotopies describing  $A_\infty$  or  $E_\infty$  structure for the multiplication map of an algebra. In this article we study bimodules over operads, which serve to encode the deep algebraic structure of *morphisms*.

The structure of the categories **DASH** and **DCSH**, first defined by Gugenheim and Munkholm in the early 1970’s [5], motivates our work. The objects of these categories have a relatively simple algebraic description, while that of the morphisms is rich and complex. The objects of **DASH** are coaugmented, associative chain algebras, and a morphism from  $A$  to  $A'$  is a map of chain coalgebras  $B(A) \rightarrow B(A')$ , where  $B$  denotes the bar construction. Dually, the objects of **DCSH** are augmented, coassociative chain coalgebras, and a morphism from  $C$  to  $C'$  is a map of chain algebras  $\Omega C \rightarrow \Omega C'$ , where  $\Omega$  denotes the cobar construction. A chain map  $f$  between two chain coalgebras  $C$  and  $C'$  is a *DCSH map* if there is an algebra morphism  $g : \Omega C \rightarrow \Omega C'$  such that the linear part of  $g$  is  $s^{-1}f$ . There is a dual definition of *DASH maps*.

The categories **DASH** and **DCSH** play an important role in topology. Let  $C_*$  denote the normalized chains functor from simplicial sets to chain complexes. Let  $K$  be any reduced simplicial set, and let  $GK$  be the Kan loop group on  $K$ . The usual coproduct on  $C_*K$  is a *DCSH map*. Moreover, as shown in [7], there is a natural coproduct on  $\Omega C_*K$  with respect to which the natural equivalence of chain

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algebras  $\Omega C_*K \rightarrow C_*GK$  defined by Szczarba [15] is also DCSH map. Among the applications to rational homotopy theory, Bousfield and Gugenheim showed that the Stokes map  $A_{PL}(X) \xrightarrow{\sim} C^*(X, \mathbf{Q})$ , a weak equivalence that allows the passage to the commutative algebra category, is a DASH map [3]. In addition, Stasheff and Halperin have exploited DASH maps to study the collapse of the Eilenberg-Moore spectral sequence [14].

As an application of the general theory we develop here, we obtain a purely operadic characterization of the categories **DASH** and **DCSH**, in terms of an explicit bimodule  $\mathcal{F}$  over the associative operad  $\mathcal{A}$  in the category of chain complexes. Furthermore, the bimodule  $\mathcal{F}$  is a minimal, free  $\mathcal{A}$ -bimodule resolution of  $\mathcal{A}$  itself and therefore an important tool for studying the homological algebra of  $\mathcal{A}$ -modules.

Our approach to characterizing morphisms in terms of bimodules over operads can be summarized as follows. Let  $(\mathbf{M}, \otimes, I)$  be a closed, bicomplete, symmetric monoidal category. Let  $\Sigma_n$  denote the symmetric group on  $n$  letters. Let  $\Sigma$  denote the symmetric groupoid, i.e., the objects of  $\Sigma$  are the non-negative integers and  $\Sigma(n, m)$  is  $\Sigma_n$  if  $n = m \geq 1$ , is the trivial group if  $n = m = 0$  and is the empty set otherwise. Consider the category  $\mathbf{M}^\Sigma$  of symmetric sequences in  $\mathbf{M}$ , which can be seen as functors from  $\Sigma^{\text{op}}$  to  $\mathbf{M}$  or as sequences  $\mathcal{X} = \{\mathcal{X}(n) \mid n \geq 1\}$  of objects in  $\mathbf{M}$  such that  $\mathcal{X}(n)$  admits a right action of  $\Sigma_n$  for all  $n$ .

The category  $\mathbf{M}^\Sigma$  admits three distinct monoidal structures: the *level* structure  $(\mathbf{M}^\Sigma, \otimes, \mathcal{C})$ , the *graded* structure  $(\mathbf{M}^\Sigma, \odot, \mathcal{U})$ , and the *composition* structure  $(\mathbf{M}^\Sigma, \circ, \mathcal{J})$ . The level and graded monoidal structures are both symmetric and closed. The composition structure, however, is not symmetric and is only right closed. The name of the composition structure is justified by the fact that there is a monoidal functor from  $\mathbf{M}^\Sigma$  to the category of endofunctors on  $\mathbf{M}$  under composition; see [13].

Given the composition monoidal structure, it is easy to define operads, their (bi)modules and their (co)algebras. An *operad* is a composition monoid  $(\mathcal{P}, \gamma, \eta)$ , i.e., a symmetric sequence  $\mathcal{P}$  endowed with an associative multiplication  $\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$  admitting a unit  $\eta : \mathcal{J} \rightarrow \mathcal{P}$ . A *left  $\mathcal{P}$ -module* consists of a symmetric sequence  $\mathcal{M}$  endowed with a left action  $\lambda : \mathcal{P} \circ \mathcal{M} \rightarrow \mathcal{M}$  satisfying the usual associativity and unit axioms. *Right  $\mathcal{P}$ -modules*  $(\mathcal{M}, \rho)$  are defined analogously, and  $(\mathcal{P}, \mathcal{Q})$ -*bimodules*  $(\mathcal{R}, \lambda, \rho)$  are symmetric sequences endowed with commuting left and right actions of  $\mathcal{P}$  and  $\mathcal{Q}$ . We denote the categories of right and left  $\mathcal{P}$ -modules by  $\mathbf{Mod}_{\mathcal{P}}$  and  ${}_{\mathcal{P}}\mathbf{Mod}$ .

The definition of algebras and coalgebras over a given operad  $\mathcal{P}$  is somewhat less concise. A  *$\mathcal{P}$ -algebra* is an object  $A$  of  $\mathbf{M}$ , together with a set of equivariant morphisms in  $\mathbf{M}$

$$\{\theta_n : \mathcal{P}(n) \otimes A^{\otimes n} \rightarrow A \mid n \geq 0\},$$

where  $\Sigma_n$  acts on  $A^{\otimes n}$  by permuting factors, and commuting appropriately with the composition product  $\gamma$  on  $\mathcal{P}$ . Dually, a  *$\mathcal{P}$ -coalgebra* consists of an object  $C$  of  $\mathbf{M}$ , together with a set of equivariant morphisms in  $\mathbf{M}$

$$\{\theta_n : C \otimes \mathcal{P}(n) \rightarrow C^{\otimes n} \mid n \geq 0\}$$

commuting appropriately with the composition product  $\gamma$ . The categories of  $\mathcal{P}$ -algebras and  $\mathcal{P}$ -coalgebras are denoted  $\mathcal{P}\text{-Alg}$  and  $\mathcal{P}\text{-Coalg}$ .

Crucial to our characterization of morphisms is the observation that the categories of  $\mathcal{P}$ -algebras and of  $\mathcal{P}$ -coalgebras embed in the categories of left  $\mathcal{P}$ -modules

and of right  $\mathcal{P}$ -modules. Questions concerning *duality* of algebras and coalgebras can therefore be viewed as questions of *chirality* of right and left modules, thanks to the asymmetry of the composition product.

The embeddings are defined as follows. Let  $O \in \mathbf{M}$  be the initial object. We define two embeddings of  $\mathcal{P}$ -algebras as left  $\mathcal{P}$ -modules. Given a  $\mathcal{P}$ -algebra  $A$ , the *constant symmetric sequence*  $c(A)$  has  $c(A)(n) = A$  for all  $n > 0$  and  $c(A)(0) = O$ . The *trivial symmetric sequence*  $z(A)$  has  $z(A)(0) = A$  and  $z(n) = O$  for  $n \geq 1$ . Both  $c(A)$  and  $z(A)$  are naturally left  $\mathcal{P}$ -modules. On the other hand, if  $C$  is a  $\mathcal{P}$ -coalgebra, then the *free symmetric monoid*  $\mathcal{T}(C)$ , with  $\mathcal{T}(C)(n) = C^{\otimes n}$  for all  $n > 0$  and  $\mathcal{T}(C)(0) = I$ , admits a natural right  $\mathcal{P}$ -module structure. The algebra embedding  $c$  is well-known (cf. Kapranov-Manin, [8]), and, while we suspect that the algebra embedding  $z$  and the coalgebra embedding are a part of operad folklore, we were not able to find them in print.

There are numerous other sources of (bi)modules over operads. For example, a morphism of operads  $\mathcal{P} \rightarrow \mathcal{Q}$  endows  $\mathcal{Q}$  with the structure of a  $\mathcal{P}$ -bimodule. Moreover, if  $\mathcal{X}$  is any symmetric sequence, then  $\mathcal{P} \circ \mathcal{X} \circ \mathcal{Q}$  is naturally a  $(\mathcal{P}, \mathcal{Q})$ -bimodule. Finally, as Ching [4] and McCarthy and Minasian [10] recently showed, the functor calculus is a rich source of (bi)modules over operads. Ching proved that the derivatives of the identity functor on based spaces form an operad  $\partial_* I$  and that any based space naturally gives rise to a right  $\partial_* I$ -module and to a left  $\partial_* I$ -module. On the other hand, McCarthy and Minasian explained how to construct an operad  $\mathbf{a}_F$  from any triple  $(\text{monad})(F, \mu, \eta)$  on the category of  $S$ -modules and showed that if the Goodwillie tower of  $F(X)$  splits for every spectrum  $X$ , then  $F(X)$  is an  $\mathbf{a}_F$ -algebra, i.e.,  $c(F(X))$  is a left  $\mathbf{a}_F$ -module.

The influence of bimodules over operads on morphisms manifests itself in many ways. We can, for example, consider categories of morphisms governed by a given bimodule. Given a right  $\mathcal{P}$ -module  $\mathcal{M}$  and a left  $\mathcal{P}$ -module  $\mathcal{N}$ , the composition product of  $\mathcal{M}$  and  $\mathcal{N}$  over  $\mathcal{P}$ , denoted  $\mathcal{M} \circ_{\mathcal{P}} \mathcal{N}$ , is the symmetric sequence defined in the obvious manner. To a  $(\mathcal{P}, \mathcal{Q})$ -bimodule  $(\mathcal{R}, \lambda, \rho)$  are associated two comma categories  $(- \circ_{\mathcal{P}} \mathcal{R}) \downarrow \mathbf{Mod}_{\mathcal{Q}}$  and  $(\mathcal{R} \circ_{\mathcal{Q}} -) \downarrow {}_{\mathcal{P}}\mathbf{Mod}$ , the objects of which are  $\mathcal{R}$ -relative morphisms of right  $\mathcal{Q}$ -modules

$$\mathcal{M} \circ_{\mathcal{P}} \mathcal{R} \rightarrow \mathcal{N}$$

and of left  $\mathcal{P}$ -modules

$$\mathcal{R} \circ_{\mathcal{Q}} \mathcal{N} \rightarrow \mathcal{M},$$

respectively. By restricting to the full subcategories of  $\mathcal{Q}$ -coalgebras and of  $\mathcal{P}$ -algebras, we obtain the categories

$$(- \circ_{\mathcal{P}} \mathcal{R}) \downarrow \mathcal{Q}\text{-Coalg}$$

of  $\mathcal{R}$ -morphisms of  $\mathcal{Q}$ -coalgebras and

$$(\mathcal{R} \circ_{\mathcal{Q}} -) \downarrow \mathcal{P}\text{-Alg}$$

of  $\mathcal{R}$ -morphisms of  $\mathcal{P}$ -algebras.

Given a  $\mathcal{P}$ -bimodule  $\mathcal{R}$  that is a comonoid with respect to  $\circ_{\mathcal{P}}$ , we can use the notion of  $\mathcal{R}$ -morphism to “fatten up” the categories of left and right  $\mathcal{P}$ -modules, as well as those of  $\mathcal{P}$ -algebras and  $\mathcal{P}$ -coalgebras, leaving the objects fixed but expanding the morphism sets. As we explain below, the categories **DASH** and

**DCSH** are “fattened” versions of  $\mathcal{A}$ -**Alg** and  $\mathcal{A}$ -**Coalg**, where  $\mathbf{M}$  is the category of chain complexes.

Let  $\psi : \mathcal{R} \rightarrow \mathcal{R} \circ_{\mathcal{P}} \mathcal{R}$  denote the coproduct on  $\mathcal{R}$ , which is coassociative and counital with respect to a  $\mathcal{P}$ -bimodule morphism  $\varepsilon : \mathcal{R} \rightarrow \mathcal{I}$ . We then define the fattened categories  $(\mathcal{P}, \psi)\mathbf{Mod}$ ,  $\mathbf{Mod}_{(\mathcal{P}, \psi)}$ ,  $(\mathcal{P}, \psi)\mathbf{-Alg}$ , and  $(\mathcal{P}, \psi)\mathbf{-Coalg}$  to have the same objects as their thinner counterparts, but to have morphisms given by

$$(\mathcal{P}, \psi)\mathbf{Mod}(\mathcal{M}, \mathcal{N}) := \{\theta : \mathcal{R} \circ_{\mathcal{P}} \mathcal{M} \rightarrow \mathcal{N} \mid \theta \text{ a morphism of left } \mathcal{P}\text{-modules}\};$$

and

$$(\mathcal{P}, \psi)\mathbf{-Alg}(A, A') := (\mathcal{P}, \psi)\mathbf{Mod}(c(A), c(A'));$$

similarly for right  $\mathcal{P}$ -modules and  $\mathcal{P}$ -coalgebras. Note that  $\mathcal{P}\mathbf{Mod}(\mathcal{M}, \mathcal{N})$  embeds naturally in  $(\mathcal{P}, \psi)\mathbf{Mod}(\mathcal{M}, \mathcal{N})$ , by sending  $\varphi$  to

$$\varepsilon \circ_{\mathcal{P}} \varphi : \mathcal{R} \circ_{\mathcal{P}} \mathcal{M} \rightarrow \mathcal{P} \circ_{\mathcal{P}} \mathcal{N} \cong \mathcal{N}.$$

Similar embeddings exist of the “strict” right module, algebra and coalgebra categories into their “fattened” versions.

The composition of morphisms in these categories is defined in terms of  $\psi$ . Given  $\theta \in (\mathcal{P}, \psi)\mathbf{Mod}(\mathcal{M}, \mathcal{M}')$  and  $\theta' \in (\mathcal{P}, \psi)\mathbf{Mod}(\mathcal{M}', \mathcal{M}'')$ , their composite  $\theta'\theta \in (\mathcal{P}, \psi)\mathbf{Mod}(\mathcal{M}, \mathcal{M}'')$  is given by composing the following sequence of (strict) morphisms of left  $\mathcal{P}$ -modules.

$$\mathcal{R} \circ_{\mathcal{P}} \mathcal{M} \xrightarrow{\psi \circ 1_{\mathcal{M}}} \mathcal{R} \circ_{\mathcal{P}} \mathcal{R} \circ_{\mathcal{P}} \mathcal{M} \xrightarrow{1_{\mathcal{R}} \circ \theta} \mathcal{R} \circ_{\mathcal{P}} \mathcal{M}' \xrightarrow{\theta'} \mathcal{M}''$$

Composition in  $\mathbf{Mod}_{(\mathcal{P}, \psi)}$  is defined similarly, while composition in  $(\mathcal{P}, \psi)\mathbf{-Alg}$  and in  $(\mathcal{P}, \psi)\mathbf{-Coalg}$  is obtained by restriction from  $(\mathcal{P}, \psi)\mathbf{Mod}$  and  $\mathbf{Mod}_{(\mathcal{P}, \psi)}$ . We call  $(\mathcal{P}, \psi)\mathbf{Mod}$ ,  $\mathbf{Mod}_{(\mathcal{P}, \psi)}$ ,  $(\mathcal{P}, \psi)\mathbf{-Alg}$ , and  $(\mathcal{P}, \psi)\mathbf{-Coalg}$  the  $(\mathcal{R}, \psi)$ -governed versions of their strict counterparts.

The reader familiar with category theory will have recognized that the  $\mathcal{R}$ -governed categories of left and right  $\mathcal{P}$ -modules are precisely the Kleisli categories associated to the comonads  $\mathcal{R} \circ_{\mathcal{P}} -$  and  $- \circ_{\mathcal{P}} \mathcal{R}$ , respectively.

A plentiful source of composition comonoids is thus essential to producing “fattened” categories of modules over an operad. We have developed a tool for constructing composition comonoids over the associative operad  $\mathcal{A}$ , when  $\mathbf{M}$  is the category of chain complexes over a commutative ring  $R$ . Let  $\mathbf{Csg}_{\otimes}$  denote the category of level cosemigroups in  $\mathbf{M}^{\Sigma}$ , i.e., of symmetric sequences  $\mathcal{X}$  endowed with a coassociative level comultiplication  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \otimes \mathcal{X}$  that is not necessarily counital. We define a functor, called the *diffracting functor*,

$$\Phi : \mathbf{Csg}_{\otimes} \rightarrow \mathcal{A}\mathbf{Mod}_{\mathcal{A}}$$

such that  $\Phi(\mathcal{X})$  is naturally a comonoid with respect to  $\circ_{\mathcal{A}}$ , for all level cosemigroups  $\mathcal{X}$ . As a consequence, there exist  $\Phi(\mathcal{X})$ -governed versions of the categories of left and right  $\mathcal{A}$ -modules and of  $\mathcal{A}$ -algebras and  $\mathcal{A}$ -coalgebras. Let  $\psi_{\mathcal{X}} : \Phi(\mathcal{X}) \rightarrow \Phi(\mathcal{X}) \circ_{\mathcal{A}} \Phi(\mathcal{X})$  denote the natural comultiplication.

Chain complex variations on the suspension operad of Arone, Bauer, Johnson and Morava [1] are vital ingredients in the definition of the functor  $\Phi$ . Let  $\mathcal{S}$  denote the sequence of chain complexes such that for all  $n$ ,  $\mathcal{S}(n) = R\{s_{n-1}\}$ , the free  $R$ -module on a generator of degree  $n - 1$ . Let  $\mathcal{S}_{\chi}$  denote  $\mathcal{S}$  endowed with the

sign action of  $\Sigma_n$  in level  $n$ . Let  $\mathcal{S}_\Sigma = \mathcal{S} \otimes \mathcal{A}$ , i.e.,  $\mathcal{S}_\Sigma(n) = \mathcal{S}(n) \otimes R[\Sigma_n]$ , with the free right  $\Sigma_n$ -action. As a symmetric sequence of graded modules,  $\Phi(\mathcal{X})$  is  $\mathcal{A} \circ (\mathcal{S}_\chi \otimes \mathcal{X}) \circ \mathcal{S}_\Sigma \circ \mathcal{A}$ . The level-wise differential on  $\Phi(\mathcal{X})$  is expressed in terms of natural simplicial and precosimplicial structure.

The interplay between the level monoidal structure and the diffracting functor is worth examining as well. There is a natural transformation, which we call the *generalized Milgram map*,

$$q : \Phi(- \otimes -) \Rightarrow \Phi(-) \otimes \Phi(-)$$

of functors from  $\mathbf{Csg}_\otimes^{\times 2}$  to  $\mathcal{A}\mathbf{Mod}_\mathcal{A}$ . The transformation  $q$  is a sort of generalized Alexander-Whitney map, along the same lines as the original Milgram map  $\Omega(C \otimes C') \rightarrow \Omega C \otimes \Omega C'$  for chain coalgebras  $C$  and  $C'$  [11]. Consequently, if the level comultiplication  $\Delta$  on  $\mathcal{X}$  is itself a morphism of level cosemigroups, then  $\Phi(\mathcal{X})$  naturally has the structure of a level cosemigroup, with comultiplication  $q\Phi(\Delta)$ .

The diffracting functor satisfies a beautiful duality relation, with respect to both the cobar construction and the bar construction. Let  $\mathcal{X}$  be a level cosemigroup. If  $\mathcal{Y}$  and  $\mathcal{Z}$  are level (co)semigroups, then a morphism  $\mathcal{Y} \circ \mathcal{X} \rightarrow \mathcal{Z}$  is called *(co)multiplicative* if it is compatible with the (co)multiplications in  $\mathcal{Y}$  and  $\mathcal{Z}$  and with the comultiplication in  $\mathcal{X}$ .

**Bar Duality Theorem.** *Let  $(\mathcal{X}, \Delta)$  be any level cosemigroup. Let  $F, G : \mathbf{D} \rightarrow \mathcal{A}\mathbf{Alg}$  be functors, where  $\mathbf{D}$  is any category. Let  $\mathbf{B}_\mathcal{X}(F, G)$  be the set of comultiplicative natural transformations  $\tau : \mathcal{X} \circ c(BF) \rightarrow c(BG)$  of functors from  $\mathbf{D}$  to  $\mathbf{Mod}_\mathcal{A}$ . Let  $\mathbf{\bar{C}}_\mathcal{X}(F, G)$  be the set of natural transformations  $\tau : \Phi(\mathcal{X}) \circ c(F) \rightarrow c(G)$  of functors from  $\mathbf{D}$  to  $\mathcal{A}\mathbf{Mod}$ . Then there exist natural bijections*

$$\mathbf{B}_\mathcal{X}(F, G) \cong \mathbf{\bar{C}}_\mathcal{X}(F, G).$$

Let  $A$  and  $A'$  be any two associative chain algebras. According to the Bar Duality Theorem, an  $\mathcal{X}$ -relative morphism  $BA \xrightarrow{\mathcal{X}} BA'$  can be obtained by suspending and then comultiplicatively lifting a family

$$\{\Phi(\mathcal{X})(m) \otimes A^{\otimes m} \rightarrow A' \mid m \geq 1\}$$

of appropriately equivariant morphisms of chain complexes. On the other hand, by restriction and desuspension, an  $\mathcal{X}$ -relative morphism  $BA \xrightarrow{\mathcal{X}} BA'$  gives rise to a  $\Phi(\mathcal{X})$ -relative morphism of chain algebras  $A \xrightarrow{\Phi(\mathcal{X})} A'$ .

As an application of Bar Duality, we can prove the existence of natural  $\mathcal{X}$ -governed morphisms between bar constructions by acyclic models methods, since  $\Phi(\mathcal{X})$  has a natural differential filtration.

**Theorem 0.1** (Existence of  $\mathcal{X}$ -governed morphisms I). *Let  $F$  and  $G$  be as in the Bar Duality Theorem. Let  $U : \mathcal{A}\mathbf{Alg} \rightarrow \mathbf{M}$  be the forgetful functor. If there is a set of models in  $\mathbf{D}$  with respect to which  $F$  is free and  $G$  is acyclic, then for all level cosemigroups  $(\mathcal{X}, \Delta)$  under  $\mathcal{J}$  and all natural transformations of functors  $\tau : UF \rightarrow UG$ , there exists a comultiplicative natural transformation*

$$\hat{\tau}_\mathcal{X} : \mathcal{X} \circ c(BF) \rightarrow c(BG)$$

*lifting the suspension of  $\tau$ .*

The cobar version of the duality theorem requires a certain permutation condition on the natural transformations considered.

**Cobar Duality Theorem.** *Let  $(\mathcal{X}, \Delta)$  be any level cosemigroup. Let  $F, G : \mathbf{D} \rightarrow \mathcal{A}\text{-Coalg}$  be functors, where  $\mathbf{D}$  is a small category. Let  $\Omega_{\mathcal{X}}(F, G)$  be the set of multiplicative natural transformations  $\tau : \mathcal{T}(\Omega F) \circ \mathcal{X} \rightarrow \mathcal{T}(\Omega G)$  of functors from  $\mathbf{D}$  to  $\mathcal{A}\text{Mod}$ . Let  $\mathfrak{U}_{\mathcal{X}}(F, G)$  be the set of transposed tensor natural transformations  $\tau : \mathcal{T}(F) \circ \Phi(\mathcal{X}) \rightarrow \mathcal{T}(G)$  of functors from  $\mathbf{D}$  to  $\text{Mod}_{\mathcal{A}}$ . Then there exist natural bijections*

$$\Omega_{\mathcal{X}}(F, G) \cong \mathfrak{U}_{\mathcal{X}}(F, G).$$

Let  $C$  and  $C'$  be any two coassociative chain coalgebras. The Cobar Duality Theorem implies that an  $\mathcal{X}$ -relative morphism  $\Omega C \xrightarrow{\mathcal{X}} \Omega C'$  can be obtained by desuspending and then multiplicatively extending a family

$$\{C \otimes \Phi(\mathcal{X})(m) \rightarrow (C')^{\otimes m} \mid m \geq 1\}$$

of appropriately equivariant morphisms of chain complexes. On the other hand, by restriction and suspension, an  $\mathcal{X}$ -relative morphism  $\Omega C \xrightarrow{\mathcal{X}} \Omega C'$  gives rise to a  $\Phi(\mathcal{X})$ -relative morphism of chain coalgebras  $C \xrightarrow{\Phi(\mathcal{X})} C'$ .

Acyclic models methods again permit us to establish the existence of natural  $\mathcal{X}$ -governed morphisms, now between cobar constructions.

**Theorem 0.2** (Existence of  $\mathcal{X}$ -governed morphisms II). *Let  $F$  and  $G$  be as above. Let  $U' : \mathcal{A}\text{-Coalg} \rightarrow \mathbf{M}$  be the forgetful functor. If there is a set of models in  $\mathbf{D}$  with respect to which  $F$  is free and  $G$  is acyclic, then for all level cosemigroups  $(\mathcal{X}, \Delta)$  and for all natural transformations  $\tau : U'F \rightarrow U'G$ , there exists a multiplicative natural transformation*

$$\widehat{\tau}_{\mathcal{X}} : \mathcal{T}(\Omega F) \circ \mathcal{X} \rightarrow \mathcal{T}(\Omega G)$$

extending the desuspension of  $\tau$ .

We now examine in detail the result of applying  $\Phi$  to the composition unit  $\mathcal{J}$ . Let  $\mathcal{F} = \Phi(\mathcal{J}) = \mathcal{A} \circ \mathcal{S}_{\Sigma} \circ \mathcal{A}$ , which we call the *Alexander-Whitney  $\mathcal{A}$ -bimodule*. Recall that  $\psi_{\mathcal{J}} : \mathcal{F} \rightarrow \mathcal{F} \circ_{\mathcal{A}} \mathcal{F}$  denotes the canonical composition comultiplication on  $\mathcal{F}$ .

Note that  $\mathcal{F}$  admits the structure of a level cosemigroup, since the level comultiplication  $\Delta_{\mathcal{J}}$  on  $\mathcal{J}$  is necessarily a morphism of level cosemigroups. We can therefore apply the generalized Milgram transformation to obtain a level comultiplication  $\Delta_{\mathcal{F}} = q\Phi(\Delta_{\mathcal{J}})$  on  $\mathcal{F}$ .

The following theorem summarizes the most important properties of  $\mathcal{F}$ .

**Alexander-Whitney Theorem.** *Let  $\mathcal{F} = \Phi \mathcal{J}$ .*

- (1) **DASH** =  $(\mathcal{A}, \psi_{\mathcal{J}})\text{-Alg}$  and **DCSH** =  $(\mathcal{A}, \psi_{\mathcal{J}})\text{-Coalg}$ .
- (2) The natural morphism  $\mathcal{F} \rightarrow \mathcal{A}$  of  $\mathcal{A}$ -bimodules is a quasi-isomorphism in positive levels, i.e.,  $\mathcal{F}$  is a free  $\mathcal{A}$ -bimodule resolution of  $\mathcal{A}$ .
- (3) Let  $F, G : \mathbf{D} \rightarrow \mathcal{A}\text{-Alg}$  be functors, where  $\mathbf{D}$  is a category admitting a set of models with respect to which  $F$  is free and  $G$  is acyclic. Let  $U : \mathcal{A}\text{-Alg} \rightarrow \mathbf{M}$  be the forgetful functor. For any natural transformation  $\tau : UF \rightarrow UG$ , there exists a natural transformation of functors into chain coalgebras  $\widehat{\tau}_{\mathcal{J}} : BF \rightarrow BG$  lifting the suspension of  $\tau$ .
- (4) Let  $F, G : \mathbf{D} \rightarrow \mathcal{A}\text{-Coalg}$  be functors, where  $\mathbf{D}$  is a small category admitting a set of models with respect to which  $F$  is free and  $G$  is acyclic. Let

$U' : \mathcal{A}\text{-Coalg} \rightarrow \mathbf{M}$  be the forgetful functor. For any natural transformation  $\tau : U'F \rightarrow U'G$ , there exists a natural transformation of functors into chain algebras  $\widehat{\tau}_{\mathcal{J}} : \Omega F \rightarrow \Omega G$  extending the desuspension of  $\tau$ .

The first property asserts that we have attained our goal of providing a purely operadic description of the categories **DASH** and **DCSH** and follows immediately from the Bar and Cobar Duality Theorems, applied to  $\mathcal{X} = \mathcal{J}$ . The second property indicates that  $\mathcal{F}$  is a handy tool for studying the homological algebra of  $\mathcal{A}$ -modules.

We have used  $\mathcal{F}$  to provide an example of a chain coalgebra whose cohomology algebra is realizable as an algebra over the Steenrod algebra, but which is not of the homotopy type of the chain complex of any space. Consider the coassociative chain coalgebra  $M = (\mathbf{Z}_2\{1_0, u_2, x_3, y_3, z_3, v_4, w_6\}, \partial, \psi)$ , where subscript indicates degree. The only non-zero differential in  $M$  is  $\partial(v) = x + y$ . All elements other than  $v$  and  $w$  are primitive, while  $\bar{\psi}(v) = u \otimes u$  and  $\bar{\psi}(w) = x \otimes z + z \otimes y$ . It is easy to check that the mod 2 cohomology of the dual of  $M$  is isomorphic to  $H^*(S^2 \vee (S^3 \times S^3); \mathbf{Z}_2)$  as unstable algebras over the mod 2 Steenrod algebra. On the other hand there is no morphism  $\Psi : \mathcal{T}(M) \circ \mathcal{F} \rightarrow \mathcal{T}(M \otimes M)$  such that  $\Psi_1 = \psi$ , i.e.,  $\psi$  is not a DCSH morphism and therefore there is no space  $X$  such that  $M$  has the homotopy type of  $C_*(X; \mathbf{Z}_2)$ .

The final application of the diffracting functor that we treat in this article is the unification of the usual bar and cobar constructions, both of which turn out to be “shadows” cast by the diffracting functor.

**Theorem 0.3** (Plato’s Cave Theorem). [12] *Let  $\mathcal{F} = \Phi \mathcal{J}$ .*

(1) *Let  $A$  be any associative chain algebra. Then*

$$B(A, A) \cong \left( \mathcal{J} \circ_{\mathcal{A}} \mathcal{F} \circ_{\mathcal{A}} z(A) \right) (0).$$

(2) *Let  $C$  be any coassociative chain algebra. Let  $i(C)$  be the symmetric sequence with  $i(C)(1) = C$ , while  $i(C)(n) = 0$  if  $n \neq 1$ . Then*

$$\Omega C \cong \left( \mathcal{A} \circ i(C) \circ_{\mathcal{A}} \mathcal{F} \circ_{\mathcal{A}} \mathcal{J} \right) (0).$$

In view of the above theorem, we propose the following generalization of the bar and cobar constructions. If  $\mathcal{M}$  and  $\mathcal{N}$  are left and right  $\mathcal{A}$ -modules, respectively, we define

$$B(\mathcal{M}) = \mathcal{J} \circ_{\mathcal{A}} \mathcal{F} \circ_{\mathcal{A}} \mathcal{M} \cong \mathcal{I}_{\Sigma} \circ \mathcal{M}$$

and

$$\Omega(\mathcal{N}) = \mathcal{N} \circ_{\mathcal{A}} \mathcal{F} \circ_{\mathcal{A}} \mathcal{J} \cong \mathcal{N} \circ \mathcal{I}_{\Sigma}.$$

In future articles we will treat the numerous possible generalizations of the diffracting functor. In particular, we will define diffraction in an arbitrary triangulated category and over operads other than the associative operad.

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# DIEUDONNÉ MODULES AND $p$ -DIVISIBLE GROUPS ASSOCIATED WITH MORAVA $K$ -THEORY OF EILENBERG-MAC LANE SPACES

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(JOINT WORK WITH VICTOR BUCHSTABER)

ABSTRACT. We study the structure of the formal groups associated to the Morava  $K$ -theories of integral Eilenberg-Mac Lane spaces. The main result is that every formal group in the collection  $\{K(n)^*K(\mathbb{Z}, q), q = 2, 3, \dots\}$  for a fixed  $n$  enters in it together with its Serre dual, an analogue of a principal polarization on an abelian variety. We also identify the isogeny class of each of these formal groups over an algebraically closed field. These results are obtained with the help of the Dieudonné correspondence between bicommutative Hopf algebras and Dieudonné modules. We extend P. Goerss's results on the bilinear products of such Hopf algebras and corresponding Dieudonné modules.

## 1. INTRODUCTION

The theory of formal groups gave rise to a powerful method for solving various problems of algebraic topology thanks to fundamental works by Novikov [5] and Quillen [7]. Formal groups in topology arise when one applies a complex oriented cohomology theory to the infinite complex projective space  $\mathbb{C}P^\infty$ . However the formal groups obtained in this way are all one-dimensional and so far the rich and intricate theory of higher dimensional formal groups remained outside of the realm of algebraic topology. One could hope to get nontrivial examples in higher dimensions by applying a generalized cohomology to an  $H$ -space. For most known cohomology theories and  $H$ -spaces this hope does not come true, however there is one notable exception. Quite surprisingly, the Morava  $K$ -theories applied to integral Eilenberg-Mac Lane spaces give rise to formal groups in higher dimensions. Moreover, these formal groups are exceptionally good in the sense that they have *finite height*.

This striking result belongs to Ravenel and Wilson [8] who used it to prove the so-called Conner-Floyd conjecture. However until now there has not been a systematic study of the remarkable collection of formal groups discovered by Ravenel and Wilson. This study is our main objective in this paper.

The main tool for Ravenel and Wilson was the notion of a Hopf ring and its behaviour in spectral sequences. The definition of a Hopf ring was recently put in a conceptual framework by Goerss by introducing a suitable symmetric monoidal category for bicommutative Hopf algebras in [2]. We make substantial use of the results of this paper.

It is well-known that the most effective way to study formal groups, particularly those of finite height or, more generally,  *$p$ -divisible groups* is via the *Dieudonné functor* which associates to a formal group a module over a certain ring called the Dieudonné ring cf. [4], [1]. Goerss supplied the category of Dieudonné modules with a monoidal structure and showed that the Dieudonné functor is monoidal. We use this technique to study the structure of Ravenel-Wilson formal groups.

Our main result is that the spectrum multiplication

$$K(\mathbb{Z}/p^\nu, q) \wedge K(\mathbb{Z}/p^\nu, n - q) \rightarrow K(\mathbb{Z}/p^\nu, n)$$

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induces a kind of Poincaré duality on  $K(n)_*K(\mathbb{Z}/p^\nu, -)$  where  $K(n)$  is an appropriate version of the  $n$ th Morava  $K$ -theory. More precisely, we show that the Hopf algebras  $K(n)_*K(\mathbb{Z}/p^\nu, q)$  and  $K(n)_*K(\mathbb{Z}/p^\nu, n - q)$  are dual to each other. Moreover, the formal groups  $K(n)^*K(\mathbb{Z}, q + 1)$  and  $K(n)^*K(\mathbb{Z}, n - q + 1)$  are Serre dual to each other and we identify explicitly the isogeny classes of these formal groups over an algebraic closure of  $\mathbb{F}_p$ , the field of  $p$  elements.

The main ingredient in the proof is the theorem of Ravenel and Wilson which shows that the collection of Hopf algebras  $K(n)_*K(\mathbb{Z}/p^\nu, -)$  forms an *exterior* Hopf ring on  $K(n)_*K(\mathbb{Z}/p^\nu, 1)$ . The ‘Poincaré duality’ mentioned above is not a formal consequence of the Ravenel-Wilson theorem, though, but follows from rather exceptional properties of the Hopf algebra  $K(n)_*K(\mathbb{Z}/p^\nu, 1)$ .

## 2. $p$ -DIVISIBLE GROUPS ASSOCIATED WITH $K(n)^*K(\mathbb{Z}, q)$

**2.1. Basic definitions.** We start by recalling some standard definitions and facts from the theory of  $p$ -divisible groups referring the reader to [1], [9] or [10] for details. Here  $\mathbf{k}$  is an algebraic extension of  $\mathbb{F}_p$  containing a square root of  $-1$  and  $p > 2$ .

**Definition 2.1.** A  $p$ -divisible group of height  $h$  over a field  $\mathbb{F}_p$  is a sequence  $G = (H_\nu, i_\nu)$ ,  $\nu = 0, 1, 2, \dots$  of bicommutative Hopf algebras over  $\mathbf{k}$  with  $\dim H_\nu = \nu h$  and  $i_\nu : H_{\nu+1} \rightarrow H_\nu$  is a Hopf algebra homomorphism such that for each  $\nu$  the sequence

$$H_{\nu+1} \xrightarrow{[p^\nu]} H_{\nu+1} \xrightarrow{i_\nu} H_\nu \longrightarrow 0$$

is exact in  $\mathcal{H}$ .

Since  $[p^{\nu+1}] = [p] \cdot [p^\nu]$  is the trivial endomorphism of  $H_{\nu+1}$  we conclude that there exists a map  $j_\nu : H_\nu \rightarrow H_{\nu+1}$  making commutative the following diagram

$$\begin{array}{ccc} & H_{\nu+1} & \\ & \uparrow [p] & \swarrow j_\nu \\ & H_{\nu+1} & \xrightarrow{i_\nu} H_\nu \end{array}$$

The topological Hopf algebra  $H = \varprojlim H_\nu$  represents a formal group from which the sequence  $(H_\nu, i_\nu)$  could be recovered by setting  $H_\nu := \text{coker}\{[p^\nu] : H \rightarrow H\}$ . We will use the term ‘ $p$ -divisible group’ also for the corresponding formal group. The dimension of the  $p$ -divisible group  $G$  is the Krull dimension of  $H$ .

Next,  $H$  will be isomorphic to a ring of formal power series (one say that in this case the corresponding formal group is *smooth*) if and only if each  $H_\nu$  is a local ring.

There is a suitable version of duality for  $p$ -divisible groups.

**Definition 2.2.** If  $G = (H_\nu, i_\nu)$  is a  $p$ -divisible group then its *Serre dual*  $p$ -divisible group is defined as  $DG = (H_\nu^*, j_\nu^*)$ . It has the same height as  $(H_\nu, i_\nu)$ .

### Example 2.3.

- (1) A one-dimensional  $p$ -divisible group of height 1 is represented by the topological Hopf algebra  $H = \mathbf{k}[[x]]$  with  $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$ , the so-called multiplicative formal group. This  $p$ -divisible group is smooth. Its Serre dual  $p$ -divisible group is represented by the Hopf algebra  $\mathbf{k}[\mathbb{Z}]^*$ , the dual group ring of the infinite cyclic group. The dimension of this  $p$ -divisible group is zero and it is not smooth.
- (2) Let  $X$  be an abelian scheme over  $\mathbf{k}$ . Then the kernel of the multiplication by  $p^\nu$  on  $X$  is a finite group scheme which is represented by a Hopf algebra whose dimension is a power of  $p$ . The resulting inverse system of Hopf algebras constitutes a  $p$ -divisible group.

Dieudonné modules and  $p$ -divisible groups associated with Morava  $K$ -theory of Eilenberg-Mac Lane spaces

Two  $p$ -divisible groups are called *isogenous* if there exists a monomorphism between their representing Hopf algebras with a finite-dimensional cokernel. Two isogenous  $p$ -divisible groups have equal height.

To conclude our review of the background material we note that the height of a  $p$ -divisible group is equal to the sum of its dimension and the dimension of its dual:

$$\text{height}(G) = \text{height}(DG) = \dim(G) + \dim(DG).$$

**2.2. Main theorems.** Now let  $H_\nu(q) := K(n)^*K(\mathbb{Z}/p^\nu, q)$ . Here  $K(n)^*(-)$  denotes the  $\mathbb{Z}/2$ -graded Morava  $K$ -theory whose coefficient field  $\mathbf{k}$  contains a square root of  $-1$ . Also denote  $H_\nu^*(q) = K(n)_*K(\mathbb{Z}/p^\nu, q)$  by  $H^\nu(q)$ . The inclusion

$$\mathbb{Z}/p^\nu \longrightarrow \mathbb{Z}/p^{\nu+1}$$

induces a map of spaces

$$K(\mathbb{Z}/p^\nu, q) \longrightarrow K(\mathbb{Z}/p^{\nu+1}, q)$$

which in turn gives rise to a map of Hopf algebras  $i_\nu : H_{\nu+1}(q) \rightarrow H_\nu(q)$ .

We can formulate now our main result.

**Theorem 2.4.**

- (1) *The sequence  $(H_\nu(q), i_\nu)$  forms a  $p$ -divisible group of height  $\binom{n}{q}$ .*
- (2) *For  $q = 1, 2, \dots, n-1$  the  $p$ -divisible group  $(H_\nu(q), i_\nu)$  is smooth. The corresponding formal group is represented by a formal power series ring on  $\binom{n-1}{q-1}$  variables and could be identified with  $K(n)^*K(\mathbb{Z}, q+1)$ .*
- (3) *There is an isomorphism*

$$H^\nu(n) \cong \mathbf{k}[\mathbb{Z}/p^\nu]^*.$$

- (4) (a) *Let  $0 < q < n$ . Then the  $\circ$ -pairing*

$$H^\nu(q) \boxtimes H^\nu(n-q) \longrightarrow H^\nu(n)$$

*induces an isomorphism of the formal group of  $K(n)^*K(\mathbb{Z}, q+1)$  with the Serre dual of the formal group of  $K(n)^*K(\mathbb{Z}, n-q+1)$ . In particular the Hopf algebras  $H^\nu(q)$  and  $H_\nu(n-q)$  are isomorphic.*

*Remark 2.5.* It is curious to note that the Serre duality between  $K(n)^*K(\mathbb{Z}, q+1)$  and  $K(n)^*K(\mathbb{Z}, n-q+1)$  breaks down for  $q = 0$ . Indeed,  $K(\mathbb{Z}, 1)$  is simply the circle  $S^1$  and so  $K(n)^*K(\mathbb{Z}, 1)$  cannot give rise to a formal group. However the duality between  $K(n)_*K(\mathbb{Z}/p^\nu, 0)$  and  $K(n)_*K(\mathbb{Z}/p^\nu, n)$  continues to hold. The point is that  $K(n)^*K(\mathbb{Z}, q+1)$  can no longer be related to  $K(n)_*K(\mathbb{Z}/p^\nu, q)$  for  $q = 0$ .

*Remark 2.6.* Our results are formulated under the assumption that  $\mathbf{k}$  has characteristic  $p \neq 2$ . Note, however, that an appropriate version of Theorem 2.4 holds for  $p = 2$  as well. The point is that although in this case  $K(n)$  is not a commutative ring spectrum,  $K(n)_*K(\mathbb{Z}/2^\nu, q)$  is still a bicommutative Hopf algebra. See [3] for details.

Our next result is concerned with the identification of the isogeny class of  $K(n)^*K(\mathbb{Z}, q+1)$  and relies on the well-known correspondence between  $p$ -divisible groups and Dieudonné modules, see [1]. Let  $n, q$  be two nonnegative integers  $q, n$  and define  $n_0, q_0$  by the condition that  $\frac{q}{n} = \frac{q_0}{n_0}$  and that  $q_0, n_0$  be coprime and nonnegative. Denote by  $R_{n_0, q_0}$  the isosimple Dieudonné module of height  $n_0$  and dimension  $q_0$ . Then the following result holds.

**Theorem 2.7.** *For  $0 < q < n$  the formal group of  $K(n)^*K(\mathbb{Z}, q+1)$  is isogenous to the product of  $\frac{1}{n_0} \binom{n}{q}$  copies of the  $p$ -divisible group corresponding to the Dieudonné module  $R_{n_0, q_0}$ .*

*Remark 2.8.* One might wonder whether the formal groups corresponding to  $K(n)^*K(\mathbb{Z}, q+1)$  are *algebraicizable*, i.e. whether there exists abelian schemes of which they are formal completions. Since abelian schemes are always isogenous with their dual the same is true for their completions. This is the so-called Manin symmetry condition, see [4] or [6]. It follows that if  $n$  is odd or if  $n$  is even but  $q \neq \frac{n}{2}$  the formal group of  $K(n)^*K(\mathbb{Z}, q+1)$  cannot be algebraicized. Note that for  $n$  even the formal group of  $K(n)^*K(\mathbb{Z}, \frac{n}{2}+1)$  is *supersingular*, i.e. it is isogenous (over  $\overline{\mathbb{F}}_p$ ) to the product of copies of one-dimensional formal group of height 2.

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# CALCULUS OF FUNCTORS AND CONFIGURATION SPACES

MICHAEL CHING

## INTRODUCTION

This is a summary of a talk given at the Conference on Pure and Applied Topology on the Isle of Skye from June 21-25, 2005. The author would like to thank the organisers of the conference for a fantastic week and for the opportunity to present the following work.

We describe a relationship between Goodwillie's calculus of homotopy functors and configuration spaces. In [3], we showed that the Goodwillie derivatives of the identity functor on based spaces form an operad of spectra. Here we show that the configuration spaces of points in a parallelizable manifold form, up to suspension and homotopy, a right module over this operad (see Proposition 3.1). We then describe how this construction might be related to work of Markl, in which he shows that these configuration spaces form a right module over the Fulton-MacPherson operads  $F_m$  constructed from the compactified configuration spaces of points in  $\mathbb{R}^m$ .

We refer the reader to Goodwillie [5] for background on the calculus of homotopy functors, and to Kathryn Hess's talk at this same conference for background on operads and modules over them. In §1 below, we summarize the construction of an operad structure on the derivatives of the identity functor. In §2, we construct from a fixed based space  $X$ , a right module  $C_X$  over that operad. In §3, we use Atiyah duality to relate this module to configuration spaces for parallelizable manifolds. Finally, in §4, we recall the work of Markl, and conjecture a connection between his constructions and ours.

Much of this work is still in progress, and we warn the reader that not all the details have been fully worked out.

## 1. DERIVATIVES OF THE IDENTITY

The identity functor on based spaces, which we write

$$I : \mathbf{Top}_* \rightarrow \mathbf{Top}_*,$$

is a fundamental example in the calculus of homotopy functors. We write  $\partial_n I$  for the  $n^{\text{th}}$  derivative of  $I$ . These derivatives were calculated by Johnson [6] and Arone-Mahowald [2].

**Proposition 1.1** (Arone-Mahowald).

$$\partial_n I \simeq \text{Map}_*(K_n, \mathbb{S}),$$

where  $K_n$  denotes the partition poset complex and  $\mathbb{S}$  is the sphere spectrum. The partition poset complex, defined below, is a finite cell complex with action of the symmetric group  $\Sigma_n$ .

**Definition 1.2.** Let  $n \geq 1$  be a fixed integer and write  $K(n)$  for the poset of partitions of the set  $\{1, \dots, n\}$ , ordered by refinement. We denote by  $\hat{0}$  the minimum element in this poset, the partition consisting of singletons, and by  $\hat{1}$  the maximum element, the partition with only one piece. Let  $K_0(n) := K(n) - \hat{0}$  and  $K_1(n) := K(n) - \hat{1}$ .

The partition poset complex  $K_n$  is the following geometric realization of a simplicial set constructed from the nerves of the categories  $K(n)$ ,  $K_0(n)$  and  $K_1(n)$ :

$$K_n := \left| \frac{N_\bullet K(n)}{N_\bullet K_0(n) \cup N_\bullet K_1(n)} \right|.$$

The symmetric group  $\Sigma_n$  acts on  $K(n)$  by permuting the elements of the set  $\{1, \dots, n\}$  and this induces an action on  $K_n$ .

The connection between the partition poset complexes and the theory of operads comes via the bar construction which we now define.

**Definition 1.3.** Let  $P$  be an augmented operad in the symmetric monoidal category  $\mathcal{C}$ . The reduced simplicial bar construction on  $P$  is the simplicial symmetric sequence  $B_\bullet(P)$  given by

$$B_n(P) := \underbrace{P \circ \dots \circ P}_n$$

with face maps given by the operad composition  $P \circ P \rightarrow P$  and the augmentation  $P \rightarrow 1$ , and degeneracy maps given by the operad unit map  $1 \rightarrow P$ . (Here,  $\circ$  is the composition product of symmetric sequences and  $1$  denotes the unit symmetric sequence in  $\mathcal{C}$ .) If  $\mathcal{C}$  is enriched and tensored over topological spaces, we write  $B(P)$  for the geometric realization of  $B_\bullet(P)$ . We call this the reduced bar construction on  $P$ . It is a symmetric sequence in the category  $\mathcal{C}$ .

The partition poset complexes are a special case of this construction.

**Example 1.4.** Let  $\underline{S}^0$  be the operad in  $\mathbf{Top}_*$  (with symmetric monoidal structure given by smash product) with  $\underline{S}^0(n) := S^0$  for all  $n$  with the obvious isomorphisms for the operad compositions. Then the partition poset complexes are homeomorphic to the pieces of the bar construction on  $\underline{S}^0$ :

$$K_n \cong B(\underline{S}^0)(n).$$

The following result of [3] was also proved by Salvatore [8].

**Proposition 1.5.** *Let  $P$  be an operad in  $\mathbf{Top}_*$  with  $P(1) \cong S^0$ . Then  $B(P)$  has a natural cooperad structure.*

**Corollary 1.6.** *The partition poset complexes form a cooperad and, dually, the Goodwillie derivatives of the identity form an operad.*

*Proof.* Taking the Spanier-Whitehead duals of the cooperad structure maps on the partition poset complexes yields operad structure maps for the models of the  $\partial_n I$  of Proposition 1.1.  $\square$

## 2. MODULES OVER THE DERIVATIVES OF THE IDENTITY

Given a right, respectively left, module  $M$  over the operad  $P$  there is a bar construction with coefficients that produces a right, respectively left, comodule  $B(M)$  over the cooperad  $B(P)$ . We use this construction to get modules over the operad  $\partial_* I$ .

Let  $X$  be a based topological space. We define a right  $\underline{S}^0$ -module  $X^\wedge$  by

$$X^\wedge(n) := X^{\wedge n},$$

the  $n$ -fold smash product of  $X$ . The module structure maps take the form

$$X^{\wedge k} \wedge S^0 \wedge \dots \wedge S^0 \rightarrow X^{\wedge n_1 + \dots + n_k}$$

and are given by the reduced diagonal  $X \rightarrow X^{\wedge n_i}$  on each of the  $k$  factors of  $X$  in the source.

Applying the bar construction to this  $\underline{S}^0$ -module, we get a right comodule over the partition poset complexes:

$$M_X := B(X^\wedge)$$

and, by taking the Spanier-Whitehead dual, a right module over the derivatives of the identity:

$$C_X := \mathrm{Map}_*(M_X, \mathbb{S}).$$

This construction determines a *contravariant* functor from  $\mathbf{Top}_*$  to the category of right  $\partial_* I$ -modules.

The explicit definition of the bar construction in terms of trees allows us to identify the terms in the symmetric sequence  $M_X$  and hence  $C_X$ :

**Lemma 2.1.** *Let  $X$  be a based CW complex. With  $C_X$  defined as above, we have*

$$C_X(n) \simeq \text{Map}_*(X^{\wedge n}/\Delta, \mathbb{S})$$

where  $\Delta$  denotes the ‘fat diagonal’ in  $X^{\wedge n}$ , that is the subspace

$$\Delta := \{(x_1, \dots, x_n) \in X^{\wedge n} \mid x_i = x_j \text{ for some } i \neq j\}.$$

**Remark 2.2.** Let  $K$  be a based finite CW complex. Arone has shown in [1] that the  $n^{\text{th}}$  derivative of the functor  $X \mapsto \Sigma^\infty \text{Map}_*(K, X)$  is the spectrum

$$\text{Map}_*(K^{\wedge n}/\Delta, \mathbb{S}).$$

By Lemma 2.1, this is equivalent to  $C_K(n)$ . The collection of derivatives of this functor from based spaces to spectra thus form a right module over the derivatives of the identity.

### 3. CONFIGURATION SPACES FOR PARALLELIZABLE MODULES

In this section we analyze the module  $C_X$  when  $X$  is the one-point compactification of a parallelizable manifold  $M$ . We show that, in this case,  $C_X$  is, up to suspensions and homotopy, composed of the configuration spaces of points in the manifold  $M$ . This is just a simple consequence of Lemma 2.1 using Atiyah duality.

Let  $M$  be a parallelizable manifold and write  $M^+$  for its one-point compactification. If  $M$  is already compact then  $M^+$  stands for  $M$  with a disjoint basepoint. Since  $M^+$  is a based space we can apply the constructions of §2. The result is the following proposition.

**Proposition 3.1.** *Let  $M$  be a parallelizable  $m$ -dimensional manifold with one-point compactification  $M^+$ . Then*

$$C_{M^+}(n) \simeq \Sigma^{-mn} \Sigma^\infty C(M, n)_+$$

where  $C(M, n)_+$  denotes the configuration space of  $n$  distinct points in  $M$ , together with a disjoint basepoint. These stable configuration spaces, therefore, form a right module over the derivatives of the identity functor.

*Sketch proof.* The key point of this proof is that  $C(M, n)$  is equal to  $M^n$  minus the fat diagonal. Since  $M$  is parallelizable, this is Atiyah dual to  $M^n$  quotiented by the fat diagonal, which, by Lemma 2.1, is equivalent to  $C_{M^+}(n)$ . Alternatively, follow Nick Kuhn’s suggestion of proof by intimidation.  $\square$

## 4. CONNECTIONS WITH WORK OF MARKL

In [7], Markl shows that the configuration spaces for an  $m$ -dimensional parallelizable manifold  $M$  form a right module over the operad  $\mathbf{F}_m$ . This is the operad, described by Getzler-Jones [4], formed by the Fulton-MacPherson compactifications of the configuration spaces of points in  $\mathbb{R}^m$ . Here we suggest a possible relationship between the two right module structures, one over the derivatives of the identity and one over the Fulton-MacPherson operad.

The key to the (potential) relationship is the following construction. Let  $P$  be any operad in *unbased* topological spaces (with respect to the symmetric monoidal structure given by the cartesian product). Adding a disjoint basepoint to each of the terms in  $P$  we get an operad  $P_+$  in based spaces (with respect to the smash product). There is an obvious map of operads:

$$P_+ \rightarrow \underline{S^0}$$

given by collapsing each  $P(n)$  to the non-basepoint of  $S^0$ . Applying the bar construction and taking Spanier-Whitehead duals we get a map of operads

$$\partial_* I = \text{Map}_*(B(\underline{S^0}), \mathbb{S}) \rightarrow \text{Map}_*(B(P_+), \mathbb{S}).$$

We now take  $P$  to be equal to  $\mathbf{F}_m$ . Salvatore showed in [8] that the operad  $\mathbf{F}_m$  is 'self-dual' in the following sense:

**Lemma 4.1** (Salvatore). *Let  $\mathbf{F}_m$  denote the Fulton-MacPherson operad of compactified configuration spaces for  $\mathbb{R}^m$ . Then*

$$\text{Map}_*(B(\mathbf{F}_{m+}), \mathbb{S})(n) \simeq \Sigma^{-m(n-1)} \Sigma^\infty \mathbf{F}_m(n)_+.$$

*Moreover, these equivalences respect the operad structures on the two sides.*

Using this calculation, we obtain a map of operads

$$\partial_* I \rightarrow \Sigma^{-m} \mathbf{F}_{m+}$$

where there is a suspension spectrum understood on the right-hand side and  $\Sigma^{-m} P$  denotes the operad with

$$(\Sigma^{-m} P)(n) = \Sigma^{-m(n-1)} P(n).$$

We have no particular evidence for the following conjecture.

**Conjecture 4.2.** *The map  $\partial_* I \rightarrow \Sigma^{-m} \mathbf{F}_{m+}$  constructed above relates the right module structures formed by the configuration spaces of points in an  $m$ -dimensional parallelizable manifold over these two operads, the one given by Proposition 3.1 and the other constructed by Markl.*

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# MODULAR REPRESENTATIONS AND $E_\infty$ RING SPECTRA

DAVE BENSON

The goal of this talk is to describe recent joint work with John Greenlees [2], in which we take a problem in modular representation theory, use some machinery of Dwyer, Greenlees and Iyengar [3] to translate it into a problem about modules over  $E_\infty$  ring spectra, solve it there, and then translate the answer back again.

I don't really want to describe the actual problem until the end of the talk, because the method of translation is in some sense more interesting than the actual problem solved.

The setup is as follows. Let  $G$  be a finite group, and let  $k$  be an algebraically closed field of characteristic  $p$  dividing the order of  $G$ . We write  $\mathbf{Mod}(kG)$  for the category of  $kG$ -modules (not necessarily finitely generated) and module homomorphisms. We write  $D(kG)$  for the derived category of  $kG$ -modules, with no boundedness conditions. We write  $\mathbf{StMod}(kG)$  for the stable category of  $kG$ -modules; the objects are the same as in  $\mathbf{Mod}(kG)$ , but the arrows are given by

$$\underline{\mathbf{Hom}}_{kG}(M, N) = \mathbf{Hom}_{kG}(M, N) / \mathbf{PHom}_{kG}(M, N)$$

where  $\mathbf{PHom}_{kG}(M, N)$  is the linear subspace consisting of those homomorphisms which factor through some projective  $kG$ -module (note that the classes of projective and injective  $kG$ -modules coincide). While  $\mathbf{Mod}(kG)$  is an abelian category,  $D(kG)$  and  $\mathbf{StMod}(kG)$  are triangulated categories.

**Theorem 1.** *There is a functor  $\Phi$  of triangulated categories making the following diagram commute.*

$$\begin{array}{ccc} \mathbf{Mod}(kG) & \longrightarrow & D(kG) \\ & \searrow & \downarrow \Phi \\ & & \mathbf{StMod}(kG) \end{array}$$

*Here, the horizontal functor takes a module to the chain complex consisting of that module in degree zero and the zero module elsewhere. The diagonal functor is the obvious one.*

The idea of the proof of the theorem is to replace a complex of  $kG$ -modules by a semiinjective complex and tensor over  $k$  with a Tate resolution of  $k$  as a  $kG$ -module. The result is a Tate resolution of a module in  $\mathbf{StMod}(kG)$ .

The reader is warned that there is no such functor that preserves products and no such functor that preserves coproducts.

Next, we write  $\mathcal{R}$  for the differential graded algebra of right derived endomorphisms of  $k$  as a  $kG$ -module,

$$\mathcal{R} = \mathbb{R}\mathrm{End}_{kG}(k).$$

More explicitly, we may take a projective resolution

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$$

of the trivial  $kG$ -module  $k$ , and set

$$\mathcal{R}_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{kG}(P_i, P_{n+i})$$

with multiplication defined by composition and differential  $df$  defined so that the Leibniz rule

$$d(f(x)) = (df)(x) + (-1)^{|f|} f(dx)$$

holds. With this definition, the cycles are the chain maps, and two cycles are equivalent modulo boundaries if and only if they are chain homotopic, so the boundaries are homotopies.  $\mathcal{R}$  is a differential graded algebra with

$$H_{-n}\mathcal{R} \cong H^n(G, k).$$

**Rothenberg–Steenrod construction.** For any space  $X$  we have a quasiisomorphism (i.e., map inducing an isomorphism on homology)

$$\mathbb{R}\mathrm{End}_{C_*(\Omega X; k)}(k) \simeq C^*(X; k).$$

If  $X = BG$ , the classifying space of  $G$ , then  $\Omega X$  is homotopy equivalent to  $G$ , and so the differential graded algebra  $C_*(\Omega X; k)$  is quasiisomorphic to  $kG$  concentrated in degree zero with trivial differential. So in this case, the Rothenberg–Steenrod construction gives a quasiisomorphism

$$\mathcal{R} \simeq C^*(BG; k).$$

In general, the derived category of a differential graded algebra is defined by starting with the category of differential graded modules and homotopy classes of maps, and then inverting the quasiisomorphisms. In the case where the differential graded algebra is just a ring concentrated in degree zero, then a differential graded module is really a chain

complex of modules, and we recover the usual definition of the derived category of a ring.

It is a general fact that a quasiisomorphism of differential graded algebras gives rise to an equivalence of derived categories. So we obtain an equivalence

$$D(\mathcal{R}) \simeq D(C^*(BG; k)).$$

We assemble the objects we have described so far into the following diagram.

$$\begin{array}{ccc}
 \text{Mod}(kG) & \longrightarrow & D(kG) \xrightleftharpoons[\text{---} \otimes_{\mathcal{R}} k]{\mathbb{R}\text{Hom}_{kG}(k, -)} D(\mathcal{R}^{\text{op}}) \simeq D(C^*(BG; k)) \\
 & \searrow & \downarrow \Phi \\
 & & \text{StMod}(kG)
 \end{array}$$

The adjunction shown in the middle of this diagram is the one investigated in [3]. It gives an equivalence of categories between large subcategories of the left and the right. The idea of the game is to take problems in  $\text{StMod}(kG)$  and translate them through the diagram into problems in  $D(C^*(BG; k))$ . Note that  $\mathcal{R} \simeq \mathcal{R}^{\text{op}}$ , so the fact that  $\mathbb{R}\text{Hom}_{kG}(k, -)$  gives us right modules shouldn't bother us.

So what kind of object is  $C^*(BG; k)$ , and how does this give us any extra leverage? Well, the structure as a differential graded algebra isn't quite good enough. What we really want to exploit is the commutativity of  $C^*(BG; k)$ , and this is not so apparent at this level. But for any space  $X$ , the cochains  $C^*(X; k)$  can be regarded as an  $E_\infty$  ring spectrum via the Eilenberg–Mac Lane construction. Here,  $E_\infty$  should be thought of as meaning “associative and commutative up to all higher homotopies”. The higher associativities give rise to Massey products in cohomology, and the higher commutativities give rise to Steenrod operations in cohomology.

Our goal, then, is to treat  $D(C^*(BG; k))$  as though it were the derived category of a commutative ring, and invoke the usual techniques of commutative algebra. For this purpose, we need a category of spectra where the smash product is commutative and associative not just up to all higher homotopies, but up to coherent natural isomorphism. In the nineteen nineties, several groups of people constructed suitable categories of spectra, all with homotopy categories equivalent to the old-fashioned one. We use the version due to Elmendorf, Kříž, Mandell and May [4]; their definition of  $\mathbb{S}$ -algebras does the job, and strictly commutative  $\mathbb{S}$ -algebras are the objects corresponding to  $E_\infty$  ring spectra. The main competition at the moment is the theory of symmetric

spectra, due to Hovey, Shipley and Smith [6], though there are yet further contenders.

The main constructions from commutative algebra which we want to make use of are:

1) The tensor product of two modules over a commutative ring returns another module over the same ring. This is not at all true in non-commutative algebra. An  $E_\infty$  ring spectrum is commutative enough to make sense of tensoring two objects in the derived category and obtaining another such object.

2) In commutative algebra, one of the main tools is localisation at a prime ideal. Given a prime ideal in the homotopy of an  $E_\infty$  ring spectrum, we may localise both the ring spectrum and its derived category.

Here is a dictionary relating the left and right sides of the adjunction.

$D(kG)$	$D(C^*(BG; k))$
Induction $H \uparrow^G$ $kG \otimes_{kH} -$	Restriction via $C^*(BG; k) \rightarrow C^*(BH; k)$
Restriction $G \downarrow_H$	Coinduction $\mathbb{R}\mathrm{Hom}_{C^*(BG; k)}(C^*(BH; k), -)$
$- \otimes_k -$ with diagonal $G$ -action	$- \overset{\mathbb{L}}{\otimes}_{C^*(BG; k)} -$ with $E_\infty$ action
$k$	$C^*(BG; k)$
$kG$	$k$
Tensor product with Rickard $F$ -idempotent	Localisation
Benson–Carlson duality	Gorenstein duality

**The Gorenstein condition.**

**Theorem 2** (Benson–Carlson [1]). *If  $H^*(G, k)$  is Cohen–Macaulay then it is Gorenstein (a form of Poincaré duality; see below) with shift zero. Even if  $H^*(G, k)$  is not Cohen–Macaulay, there is a “Poincaré duality spectral sequence”.*

Greenlees [5] reformulated the Poincaré duality spectral sequence in terms of local cohomology. His version of the spectral sequence is

$$H_m^{s,t} H^*(G, k) \Rightarrow H_{-s-t}(G, k).$$

This spectral sequence is the algebraic shadow of the following theorem of Dwyer, Greenlees and Iyengar [3]]

**Theorem 3.**  $C^*(BG; k)$  is a commutative Gorenstein  $\mathbb{S}$ -algebra with shift zero.

What does this mean?

In commutative algebra, if  $R = \bigoplus_{n \geq 0} R_n$  with  $R_0 = k$  is a finitely generated graded commutative  $k$ -algebra, then  $R$  is

(i) Cohen–Macaulay if there exists a polynomial subalgebra [Noether normalisation]  $k[x_1, \dots, x_r]$  over which  $R$  is a finitely generated free module.

(ii) Gorenstein with shift  $d$  if, furthermore,  $R/(x_1, \dots, x_r)$  satisfies Poincaré duality (i.e., is finite dimensional selfinjective) with top degree  $d + \sum_{i=1}^r (|x_i| - 1)$ .

The Gorenstein condition is equivalent to the cohomological condition

$$\mathrm{Ext}_R^{i,j}(k, R) \cong \begin{cases} k & i = r \quad j = d - r \\ 0 & \text{otherwise} \end{cases}$$

Dwyer, Greenlees and Iyengar say that an  $\mathbb{S}$ -algebra  $R$  is Gorenstein with shift  $d$  if

$$\mathbb{R}\mathrm{Hom}_R(k, R) \cong k[-d]$$

in  $D(R)$ , plus some technical smallness conditions that we won't go into here. It is this definition that's involved in the theorem above.

Now recall that in commutative algebra, a localisation of a Gorenstein ring is again Gorenstein. The usual proof of this is to prove that Gorenstein is equivalent to the property that  $R$  has finite injective resolution as an  $R$ -module, and then to prove that this condition localises.

**Theorem 4** (Benson and Greenlees [2]). *If  $\mathfrak{p}$  is a homogeneous prime ideal in  $H^*(G, k)$  then the localisation  $C^*(BG; k)_{\mathfrak{p}}$  is a Gorenstein  $\mathbb{S}$ -algebra.*

The problem is that we can't use the usual proof of this from commutative algebra, because the equivalent of an injective resolution is a single object, because a module is already a complex. So having finite injective resolution doesn't make sense. So we need to find a new proof. The way we proceed is to use Grothendieck duality with respect to a

Noether normalisation coming from an embedding  $G \rightarrow SU(n)$ . We have

$$H^*(BSU(n); k) \cong k[c_2, \dots, c_n]$$

and

$$C^*(BSU(n); k) \rightarrow C^*(BG; k)$$

induces a Noether normalisation in cohomology. Grothendieck duality then says that the Matlis dual of local cohomology is Ext over the normalisation (or rather, the corresponding statement at the level of the derived category); Ext localises well and we are in good shape.

Finally, if we trace the fact that the Gorenstein condition localises through to  $\mathbf{StMod}(kG)$ , we obtain an interesting theorem. Let  $\kappa_{\mathfrak{p}}$  be the Rickard idempotent module corresponding to a prime ideal  $\mathfrak{p} \subseteq H^*(G, k)$  that picks out the layer of  $\mathbf{StMod}(kG)$  corresponding to  $\mathfrak{p}$ . So  $\kappa_{\mathfrak{p}}$  is a tensor product of an  $E$ -idempotent and an  $F$ -idempotent.

**Theorem 5** (Benson and Greenlees [2]). *As a module over  $H^*(G, k)$ , the Tate cohomology  $\hat{H}^*(G, \kappa_{\mathfrak{p}})$  is isomorphic to the injective hull of  $H^*(G, k)/\mathfrak{p}$  in the category of graded  $H^*(G, k)$ -modules, shifted in degree by minus the Krull dimension of  $H^*(G, k)/\mathfrak{p}$ .*

Consequences of this theorem include the statement that  $\kappa_{\mathfrak{p}}$  is pure injective, and that

$$\widehat{\text{Ext}}_{kG}^*(\kappa_{\mathfrak{p}}, \kappa_{\mathfrak{p}}) \cong H^*(G, k)_{\mathfrak{p}}^{\wedge} = \varprojlim_n H^*(G, k)/\mathfrak{p}^n.$$

It would be interesting to know to what extent the proofs described above can be made algebraic. Part of the point of this would be to have corresponding theorems for finite dimensional  $p$ -restricted Lie algebras, or more generally for finite group schemes.

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# THE CLASSIFICATION OF 2-COMPACT GROUPS

JESPER GRODAL

(JOINT WORK WITH KASPER K. S. ANDERSEN)

ABSTRACT. In this talk I'll announce and explain a proof of the classification of 2-compact groups, joint with K. Andersen, hence completing the classification of  $p$ -compact groups at all primes  $p$ . A  $p$ -compact group, as introduced by Dwyer-Wilkerson, is a homotopy theoretic version of a compact Lie group, but with all its structure concentrated at a single prime  $p$ . Our classification states that there is a 1-1-correspondence between connected 2-compact groups and root data over the 2-adic integers (which will be defined in the talk). As a consequence we get the conjecture that every connected 2-compact group is isomorphic to a product of the 2-completion of a compact Lie group and copies of the exotic 2-compact group  $DI(4)$ , constructed by Dwyer-Wilkerson. The major new input in the proof over the proof at odd primes (due to Andersen-Grodal-Møller-Viruel) is a thorough analysis of the concept of a root datum for 2-compact groups and its relationship with the maximal torus normalizer. With these tools in place we are able to produce a proof which to a large extent avoids case-by-case considerations.

This is a summary of my lecture given at the “Conference on Pure and Applied Topology”, Isle of Skye, Friday June 24, 2005. I announced and sketched a proof of the classification of 2-compact groups, in joint work with Kasper Andersen. This will appear in the papers [2] and [3].

Recall that a  $p$ -compact group is a triple  $(X, BX, e : X \xrightarrow{\cong} \Omega BX)$  where  $BX$  is a pointed, connected,  $p$ -complete space of the homotopy type of a  $CW$ -complex,  $X$  satisfies that  $H^*(X; \mathbb{F}_p)$  is finite over  $\mathbb{F}_p$ , and  $e$  is a homotopy equivalence. They are homotopy theoretic analogs of compact Lie groups, and were introduced by Dwyer-Wilkerson in [6]. A  $p$ -compact group is said to be connected if  $X$  is a connected space.

Our main theorem is the following

**Theorem 1.1** ([3]). *Let  $(X, BX, e : X \xrightarrow{\cong} \Omega BX)$  be a connected 2-compact group. Then*

$$BX \simeq BG_2 \times BDI(4)^s$$

where  $BG_2$  is the 2-completion of a connected compact Lie group  $G$ , and  $BDI(4)$  is the classifying space of the exotic 2-compact group  $DI(4)$  constructed by Dwyer-Wilkerson in [5],  $s \geq 0$ .

A corresponding statement for odd primes was proved by the authors together with Møller and Viruel in [4]. Partial results for  $p = 2$  have been obtained by Dwyer-Miller-Wilkerson, Notbohm, Viruel, Møller, and Vavpetič-Viruel. Our proof is a self-contained induction.

There is a better, more precise, formulation of our theorem which both makes it clear why it is the correct 2-local version of the classification of compact Lie groups and suggests a possible strategy of proof. It is based on the notion of a root datum over the  $p$ -adic integers  $\mathbb{Z}_p$ , which we now introduce.

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For  $R$  a principal ideal domain, an  $R$ -root datum  $\mathbf{D}$  is defined to be a triple  $(W, L, \{Rb_\sigma\})$ , where  $L$  is a free  $R$ -module of finite rank,  $W \subseteq \text{Aut}_R(L)$  is a finite subgroup generated by reflections (i.e., elements  $\sigma$  such that  $1 - \sigma \in \text{End}_R(L)$  has rank one), and  $\{Rb_\sigma\}$  is a collection of rank one submodules of  $L$ , indexed by the reflections  $\sigma$  in  $W$ , satisfying

$$\text{im}(1 - \sigma) \subseteq Rb_\sigma \subseteq \ker\left(\sum_{i=0}^{|\sigma|-1} \sigma^i\right) \text{ and } w(Rb_\sigma) = Rb_{w\sigma w^{-1}} \text{ for all } w \in W$$

The element  $b_\sigma \in L$ , called the *coroot* corresponding to  $\sigma$ , is determined up to a unit in  $R$ . Together with  $\sigma$  it determines a root  $\beta_\sigma : L \rightarrow R$  via the formula

$$\sigma(x) = x + \beta_\sigma(x)b_\sigma$$

If  $R = \mathbb{Z}$  then there is a 1-1-correspondence between  $\mathbb{Z}$ -root data and classically defined root data, by to  $(W, L, \{\mathbb{Z}b_\sigma\})$  associating  $(L, L^*, \{\pm b_\sigma\}, \{\pm\beta_\sigma\})$ ; see [8, Prop. 2.16]. If  $R = \mathbb{Z}$  or  $\mathbb{Z}_p$ , instead of the collection  $\{Rb_\sigma\}$  one can equivalently consider their span, the *coroot lattice*,  $L_0 = +_\sigma Rb_\sigma \subseteq L$ , and this was the definition given in [4, § 1], under the name “ $R$ -reflection datum”. Two  $R$ -root data  $\mathbf{D} = (W, L, \{Rb_\sigma\})$  and  $\mathbf{D}' = (W', L', \{Rb'_\sigma\})$  are said to be isomorphic if there exists an isomorphism  $\varphi : L \rightarrow L'$  such that  $\varphi W \varphi^{-1} = W'$  and  $\varphi(Rb_\sigma) = Rb'_{\varphi\sigma\varphi^{-1}}$ . In particular the automorphism group is given by  $\text{Aut}(\mathbf{D}) = \{\varphi \in N_{\text{Aut}_R(L)}(W) \mid \varphi(Rb_\sigma) = Rb_{\varphi(\sigma)}\}$  and we define the outer automorphism group as  $\text{Out}(\mathbf{D}) = \text{Aut}(\mathbf{D})/W$ .

We now explain how to associate a  $\mathbb{Z}_p$ -root datum to a  $p$ -compact group. By a theorem of Dwyer-Wilkerson any  $p$ -compact group  $(X, BX, e)$  has a maximal torus, which is a map  $i : BT = (BS^1_{\hat{p}})^r \rightarrow BX$  satisfying that the fiber has finite  $\mathbb{F}_p$ -cohomology and non-trivial Euler characteristic. Replacing  $i$  by an equivalent fibration, we define the *Weyl space*  $\mathcal{W}_X(T)$  as the topological monoid of self-maps  $BT \rightarrow BT$  over  $i$ . The *Weyl group* is defined as  $W_X(T) = \pi_0(\mathcal{W}_X(T))$  and the classifying space of the maximal torus normalizer is defined as  $B\mathcal{N}_X(T) = BT_h \mathcal{W}_X(T)$ . Now, by definition,  $W_X$  acts on  $L = \pi_2(BT)$  and it is a theorem of Dwyer-Wilkerson that, if  $X$  is connected, this gives a faithful representation of  $W_X$  in  $\text{Aut}_{\mathbb{Z}_p}(L)$  as a finite  $\mathbb{Z}_p$ -reflection group. There is also an easy formula for the  $b_\sigma$  in terms of the maximal torus normalizer  $\mathcal{N}_X$ , for which we refer to [8], [2], or [3]. This explains how to associate a  $\mathbb{Z}_p$ -root datum  $\mathbf{D}_X$  to a connected  $p$ -compact group  $X$ . (In the case where  $p$  is odd the root datum is in fact completely determined by the finite  $\mathbb{Z}_p$ -reflection group  $(W, L)$ , which explains the formulation of our classification theorem with Møller and Viruel in [4].)

We are now ready to state the precise version of our main theorem.

**Theorem 1.2** ([3]). *The assignment which to a connected 2-compact group  $X$  associates its  $\mathbb{Z}_2$ -root datum  $\mathbf{D}_X$  root gives a one-to-one correspondence between connected 2-compact groups and  $\mathbb{Z}_2$ -root data. Furthermore the map  $\Phi : \pi_0(\text{Aut}(BX)) \rightarrow \text{Out}(\mathbf{D}_X)$  is an isomorphism, and  $B \text{Aut}(BX)$  is the unique total space of a split fibration*

$$B^2 \mathcal{Z}(\mathbf{D}) \rightarrow B \text{Aut}(BX) \rightarrow B \text{Out}(\mathbf{D}_X)$$

Here  $\mathcal{Z}(\mathbf{D})$  is the center of the root datum  $\mathbf{D}$ , defined so as to agree with the formula for the center of a  $p$ -compact group given in [7], and  $B \text{Aut}(BX)$  denotes the classifying space of the topological monoid of self-homotopy equivalences of  $BX$ .

We remark that since we have completely described connected  $p$ -compact groups *as well as their space of automorphisms*, we also get a classification for non-connected 2-compact groups. (The situation here is totally analogous to the case of compact Lie groups.) If  $X$

is a connected  $p$ -compact group with a given root datum  $\mathbf{D}$  and  $\pi$  is a given  $p$ -group, then the  $p$ -compact groups with connected component isomorphic to  $X$  and component group  $\pi$  are in 1-1-correspondence with the  $\text{Out}(\pi)$ -orbits of the set  $[B\pi, B\text{Aut}(BX)]$ . Since both  $X$  and  $B\text{Aut}(BX)$  is completely described in terms of  $\mathbf{D}_X$  by the main theorem, this gives a classification also in the non-connected case. We refer to [3] for a more detailed description.

The main theorem has a number of corollaries. The most important is perhaps that it gives a proof of the maximal torus conjecture, giving a purely homotopy theoretic characterization of compact Lie groups amongst finite loop spaces.

**Theorem 1.3** (Maximal torus conjecture [3]). *The classifying space functor, which to a compact Lie group  $G$  associates the finite loop space  $(G, BG, e : G \xrightarrow{\cong} \Omega BG)$  gives a one-to-one correspondence between compact Lie groups and finite loop spaces with a maximal torus. (Moreover, if  $G$  is connected we have a split fibration  $B^2\mathcal{Z}(\mathbf{D}_G) \rightarrow B\text{Aut}(BG) \rightarrow B\text{Out}(\mathbf{D}_G)$ .)*

The fact that the functor “ $B$ ” is faithful was already known by work of Notbohm, Møller, and Osse and the statement about the space  $B\text{Aut}(BG)$  follows easily from earlier work of Jackowski-McClure-Oliver and Dwyer-Wilkerson. The new, and a priori quite surprising, result here is the statement that if a finite loop space has a maximal torus, then it has to come from a compact Lie group.

Another application of the classification of 2-compact groups is to give an answer to the so-called Steenrod problem for  $p = 2$  (see [13] and [12]), which asks which graded polynomial algebras can occur as the mod 2 cohomology ring of a space? Steenrod’s problem was solved for  $p$  “large enough” by Adams-Wilkerson [1] and for all odd primes by Notbohm [11] using a partial classification of  $p$ -compact groups,  $p$  odd.

**Theorem 1.4** (Steenrod’s problem for  $p = 2$  [3]). *Suppose that  $P^*$  is a graded polynomial algebra over  $\mathbb{F}_2$  in finitely many variables. If  $P^*$  occurs as  $H^*(Y; \mathbb{F}_2)$  for some space  $Y$  (no assumptions), then  $P^*$  is isomorphic, as a graded algebra, to*

$$H^*(BG; \mathbb{F}_2) \otimes H^*(BDI(4); \mathbb{F}_2)^{\otimes s} \otimes Q^*$$

where  $G$  is a connected semi-simple Lie group and  $Q^*$  is a polynomial ring with generators in degrees one and two.

In particular if  $P^*$  is assumed to have generators in degree  $\geq 3$  then  $G$  has to be simply connected and  $P^*$  is a tensor product of the following graded algebras:

$$\begin{array}{ll} \mathbb{F}_2[x_4, x_6, \dots, x_{2n}] & (\text{SU}(n)) \\ \mathbb{F}_2[x_4, x_8, \dots, x_{4n}] & (\text{Sp}(n)) \\ \mathbb{F}_2[x_4, x_6, x_7, x_8] & (\text{Spin}(7)) \\ \mathbb{F}_2[x_4, x_6, x_7, x_8, x'_8] & (\text{Spin}(8)) \\ \mathbb{F}_2[x_4, x_6, x_7, x_8, x_{16}] & (\text{Spin}(9)) \\ \mathbb{F}_2[x_4, x_6, x_7] & (G_2) \\ \mathbb{F}_2[x_4, x_6, x_7, x_{16}, x_{24}] & (F_4) \\ \mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}] & (DI(4)) \end{array}$$

It seems reasonable that one can in fact list all polynomial rings which occur as  $H^*(BG; \mathbb{F}_2)$  for  $G$  semi-simple, although we have not been able to locate such a list in the literature; for  $G$  simple a list can be found in [10].

We finally point out that many classical theorems from Lie theory by Borel, Bott, Demazure, and others also carry over to 2-compact groups via the classification. Applications of this type were already pointed out in [4], to which we refer.

We end this short summary by saying a few words about the proofs. Our strategy is an elaboration of the strategy we pursued with Møller and Viruel in [4], but with several new additions, most importantly a development and utilization of the theory of  $p$ -adic root data. The sketch below is intended to sum up the main steps without introducing too much notation. We focus on the uniqueness statement, that two connected  $p$ -compact groups with isomorphic root data are isomorphic.

It is a recent theorem of Dwyer-Wilkerson [8], extending old work of Tits, that the maximal torus normalizer  $\mathcal{N}_X$  in a 2-compact group  $X$  can be recovered from the root datum  $\mathbf{D}_X$ . However,  $\mathcal{N}_X$  has a larger group of automorphisms than  $X$ , which means that it is not the right classifying invariant.

This problem can be overcome, by also keeping track of certain “root subgroups”, a machinery which we set up in [2]. In a given maximal torus normalizer  $\mathcal{N}$ , for each reflection  $\sigma$  and coroot  $b_\sigma$  one can algebraically construct a root subgroup  $\mathcal{N}_\sigma$  of  $\mathcal{N}$ . In the setting of algebraic groups, this “root subgroup”  $\mathcal{N}_\sigma$  will be the maximal torus normalizer of  $\langle U_\alpha, U_{-\alpha} \rangle$ , where  $U_\alpha$  is the root subgroup in the sense of algebraic groups corresponding to the root  $\alpha$  dual to the coroot  $b_\sigma$ .

**Theorem 1.5** ([2]). *Let  $X$  be a connected 2-compact group. The canonical map  $\text{Out}(BX) \rightarrow \text{Out}(\mathbf{D}_X)$  factors*

$$\Phi : \text{Out}(BX) \rightarrow \text{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\}) \xrightarrow{\cong} \text{Out}(\mathbf{D}_X)$$

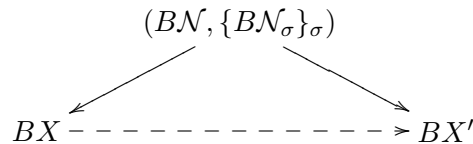
and we likewise have a canonical map

$$\Phi : B \text{Aut}(BX) \rightarrow B \text{aut}(\mathbf{D}_X)$$

where  $B \text{aut}(\mathbf{D}_X)$  is a certain space defined in [2] refining  $B \text{Aut}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$ .

These maps will be shown to be isomorphisms inductively, as part of proving Theorem 1.2. Since the pair  $(\mathcal{N}, \{\mathcal{N}_\sigma\})$  carries the same structure and automorphisms as  $\mathbf{D}$ , but is closer to  $X$ , it is more useful than  $\mathbf{D}$  for classification purposes.

Assume that we have two connected  $p$ -compact groups  $X$  and  $X'$  with the same root datum  $\mathbf{D}$  and hence the same maximal torus normalizer and root subgroups  $(\mathcal{N}, \{\mathcal{N}_\sigma\})$ , and assume by induction that Theorem 1.2 is true for all connected  $p$ -compact groups of smaller cohomological dimension. The first step is to observe that one can reduce to the case where  $X$  and  $X'$  are simple and center-free, mimicking the proofs in [4] but everywhere keeping track of the root subgroups. So we can assume that we are in the following situation



where the dotted arrow is the one that we want to construct.

Using that every element of order  $p$  can be conjugated into the maximal torus, uniquely up to conjugation in  $\mathcal{N}$ , we can by induction construct the dotted arrow on the connected

component of the centralizer of every element  $\nu : B\mathbb{Z}/p \rightarrow BX$  of order  $p$  in  $X$ .

$$\begin{array}{ccc}
 & (BC_{\mathcal{N}}(\nu)_1, \{BC_{\mathcal{N}}(\nu)_{1\sigma}\}_\sigma) & \\
 & \swarrow \qquad \qquad \searrow & \\
 BC_X(\nu)_1 & \xrightarrow{\hspace{10em}} & BC_{X'}(\nu)_1
 \end{array}$$

Using that we have control of the whole *space* of self-equivalences of  $BC_X(\nu)_1$  by Theorem 1.5 and induction, one sees that this map is equivariant with respect to the component group and hence extends uniquely to the whole centralizer  $BC_X(\nu)$ .

Now for a general elementary abelian subgroup  $\nu : E \rightarrow X$  of  $X$  we can pick an element of order  $p$  in  $E$ , and we get via restriction a map

$$BC_X(\nu) \rightarrow BC_X(\mathbb{Z}/p) \rightarrow BC_{X'}(\mathbb{Z}/p) \rightarrow BX'$$

To make sure that these maps are chosen in a compatible way, one has to show that this map does not depend on the choice of rank one subgroup of  $E$ . We developed some techniques in [4] for handling this kind of situation. Using easy (one page) case-by-case arguments on the level of 2-adic root data, we are able to verify that the assumptions of [4] are always satisfied.

Since these maps from centralizers are chosen in a compatible way, they combine to form an element

$$\vartheta \in \lim_{\nu \in \mathbf{A}(X)}^0 [BC_X(\nu), BX']$$

where  $\mathbf{A}(X)$  is the Quillen category of  $X$ . Furthermore the construction of this element allows one to show that  $X$  and  $X'$  have the same  $p$ -fusion and in particular the same maximal torus  $p$ -normalizer and the same  $\mathbb{Z}_p$ -cohomology. We are left with a rigidification question.

One approach could be to show that the relevant obstruction groups

$$\lim_{\vartheta \in \mathbf{A}(X)}^* \pi_* (\text{map}(BC_X(\nu), BX')_{[\vartheta]})$$

vanish, which would show that our diagram rigidifies and produces an equivalence

$$BX \simeq \text{hocolim}_{\nu \in \mathbf{A}(X)} BC_X(\nu) \xrightarrow{\simeq} BX'$$

It is immediate to see that this is the case for many groups  $X$  including  $DI(4)$ , and it was the approach taken for  $p$  odd in [4]. There we for instance also show that it holds for  $X = \text{PU}(n)\hat{2}$ , by the same calculation as for  $p$  odd, but it is more difficult to verify e.g., for the exceptional groups.

We however take a different approach: Since  $DI(4)$  is easily dealt with, we can assume that  $BX = BG\hat{2}$  for some simple, center-free Lie group. For each  $p$ -radical subgroup  $P$  of  $G$  we have a map  $BP \rightarrow BC_G({}_pZ(P))\hat{p} \rightarrow BX'$ . Since  $X$  and  $X'$  have the same  $\mathbb{Z}_p$ -homology, and in particular the same fundamental group, we can lift this map to a map  $B\tilde{P} \rightarrow B\tilde{X}'$ , where  $\tilde{P}$  is the preimage of  $P$  in  $\tilde{G}$ , the universal cover of  $G$ , and  $\tilde{X}'$  is the universal cover of  $X'$ .

One now verifies that these maps assemble to give an element in

$$\lim_{\tilde{G}/\tilde{P} \in \mathbf{O}_p^r(\tilde{G})}^0 [B\tilde{P}, B\tilde{X}']$$

where  $\mathbf{O}_p^r(\tilde{G})$  is the full subcategory of the  $p$ -orbit category of  $\tilde{G}$  with objects  $\tilde{G}/\tilde{P}$  for  $\tilde{P}$  a  $p$ -radical subgroup of  $\tilde{G}$ . The obstructions to rigidifying this to get a map

$$B\tilde{G} \simeq \text{hocolim}_{\tilde{G}/\tilde{P} \in \mathbf{O}_p^r(\tilde{G})} B\tilde{P} \rightarrow B\tilde{X}'$$

can be shown to lie in groups which identify with

$$\lim_{\tilde{G}/\tilde{P} \in \mathbf{O}_p^r(\tilde{G})} {}^* \pi_* (Z(\tilde{P}))$$

These groups have been shown to identically vanish by earlier work of Jackowski-McClure-Oliver [9]. Hence the map exists and it now follows easily by the construction that it is an equivalence. Passing to a quotient we get the wanted equivalence  $BG \rightarrow BX'$ , finishing the proof of the theorem.

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# CLASSES OF COMMUTATIVE SQUARES

JÉRÔME SCHERER

## INTRODUCTION

The material in this note is taken from [2], an article written with Wojciech Chachólski and Wolfgang Pitsch.

Given a map  $p : E \rightarrow B$  of spaces (i.e. simplicial sets in this note), which we do not want to assume to be a fibration, one can decompose the *global* data  $E$  into *local* pieces, namely the preimages of simplices  $\sigma : \Delta[n] \rightarrow B$  in the base. Define  $dp(\sigma)$  to fit in the following pull-back square

$$\begin{array}{ccc} dp(\sigma) & \longrightarrow & E \\ \downarrow & & \downarrow \\ \Delta[n] & \xrightarrow{\sigma} & B \end{array}$$

We stress out that this square is a pull-back square, not a homotopy pull-back square. Therefore the preimages  $dp(\sigma)$  may well have vastly different homotopy types, since  $p$  is not a fibration in general.

Now we know for example from [4, Appendix HL] that  $E$  can be reconstructed from the local pieces by means of a homotopy colimit over the simplex category  $\mathbf{B}$ . The objects of this category are the simplices of the space  $B$  and the morphisms are generated by the face and degeneracy morphisms, subject to the usual simplicial relations. We refer the reader to [3, Section 6] for more details. In fact the morphisms  $dp(\sigma) \rightarrow \Delta[n]$  form a natural transformation of diagrams indexed by  $\mathbf{B}$  and one recovers, up to homotopy, the map  $p$  by taking the homotopy colimit  $\text{hocolim}_{\mathbf{B}} dp \rightarrow \text{hocolim}_{\mathbf{B}} \Delta[n]$ .

**Question 0.1.** How closely related are the homotopy fiber  $Fib(p)$  and the preimages  $dp(\sigma)$ ?

The objective of this note is to convince the reader that the answer depends on a good understanding of the *transition functions*  $dp(\sigma) \rightarrow dp(\tau)$  induced by simplicial operations  $\theta : \Delta[n] \rightarrow \Delta[m]$ , as we illustrate next in a couple of examples.

**Example 0.2.** Let us assume that all transition functions are weak equivalences. In this case, any map  $dp(\sigma) \rightarrow Fib(p)$  is a weak equivalence as well.

This statement is a version of Puppe's Theorem [8] about homotopy colimits of natural transformations between diagrams. The simplest example is perhaps provided by push-out diagrams: Consider a map  $p$  which has been obtained as (homotopy) colimit of a natural transformation between push-out diagrams

$$\begin{array}{ccc}
 E & = & \operatorname{colim} \left( \begin{array}{ccccc} E_0 & \longleftarrow & E_1 & \hookrightarrow & E_2 \end{array} \right) \\
 p \downarrow & & \begin{array}{ccc} p_0 \downarrow & & p_1 \downarrow \\ E_0 & \longleftarrow & E_1 & \hookrightarrow & E_2 \\ B_0 & \longleftarrow & B_1 & \hookrightarrow & B_2 \end{array} \\
 B & = & \operatorname{colim} \left( \begin{array}{ccccc} B_0 & \longleftarrow & B_1 & \hookrightarrow & B_2 \end{array} \right)
 \end{array}$$

If both maps  $Fib(p_1) \rightarrow Fib(p_0)$  and  $Fib(p_1) \rightarrow Fib(p_2)$  are weak equivalences, then so is any of the three maps  $Fib(p_i) \rightarrow Fib(p)$ , for  $0 \leq i \leq 2$ .

**Example 0.3.** Let us assume that all transition functions are homology equivalences. In this case, any map  $dp(\sigma) \rightarrow Fib(p)$  is a homology equivalence as well. This is the central idea in Quillen’s group completion theorem, as exposed by McDuff and Segal [6] (and many others afterwards). Let us spend some time to explain where the transition functions come into play in the group completion, since this (and more sophisticated versions, such as Tillman’s one [9]) originated our work.

Let  $M$  be a homotopy commutative topological monoid with  $\pi_0 M \cong \mathbf{N}$ . Choose a point  $m$  in the component of 1 and construct the telescope

$$M_\infty = \operatorname{hocolim}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \dots)$$

The map  $p : EM \times_M M_\infty \rightarrow BM$  is the realization (a homotopy colimit) of a natural transformation between simplicial diagrams in which the preimages are copies of  $M_\infty$ . Because the action of  $M$  on  $M_\infty$  by *left* multiplication is by homology equivalences, we deduce that the map  $M_\infty \rightarrow Fib(p)$  is a homology equivalence as well.

Finally notice that homotopy colimits commute with themselves (Fubini rule, see for example [3, Theorem 24.9]), so that the Borel construction on the telescope  $EM \times_M M_\infty$  can be thought of as a telescope of Borel constructions

$$\operatorname{hocolim}(EM \times_M M \longrightarrow EM \times_M M \longrightarrow EM \times_M M \longrightarrow \dots),$$

which is contractible. Therefore we obtain a homology equivalence  $M_\infty \rightarrow \Omega BM$ .

Our aim is to give a unified setting to explain these two examples. The idea is that such statements have more to do with the commutative squares appearing in the natural transformations between diagrams, than with the transition functions. This allows also to give clean statements, instead of cheating (as we have done up to now in this note) in not mentioning that a map might have different homotopy fibers over distinct components.

### 1. DISTINGUISHED CLASSES

The symbol *Arrows* denotes the category whose objects are maps in *Spaces* and morphisms are commutative squares. Explicitly a morphism  $\phi : f \rightarrow g$  in *Arrows* is given by a pair  $\phi = (\phi_0, \phi_1)$  of maps for which the following square commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{\phi_0} & A \\
 f \downarrow & & \downarrow g \\
 Y & \xrightarrow{\phi_1} & B
 \end{array}$$

By a weak equivalence in *Arrows* we mean a commutative square in which both  $\phi_0$  and  $\phi_1$  are weak equivalences.

**Definition 1.1.** A class  $\mathcal{C}$  of morphisms in *Arrows* is called *distinguished* if it satisfies the four elementary closure properties:

- (1) Weak equivalences belong to  $\mathcal{C}$ .
- (2) “Homotopy invariance”. Let  $\phi : f \rightarrow g$  and  $\psi : g \rightarrow h$  be morphisms. Assume that either  $\psi$  or  $\phi$  is a weak equivalence. Then if two out of  $\phi$ ,  $\psi$ ,  $\psi\phi$  belong to  $\mathcal{C}$ , then so does the third.
- (3) “2 out of 3”. Let  $\phi : f \rightarrow g$  and  $\psi : g \rightarrow h$  be morphisms. If  $\psi$  and  $\psi\phi$  belong to  $\mathcal{C}$ , then so does  $\phi$ .
- (4) “Puppe property”. If  $F : I \rightarrow \text{Arrows}$  sends any morphism in  $I$  to a morphism in  $\mathcal{C}$ , then, for every  $i \in I$ ,  $F(i) \rightarrow \text{hocolim}_I F$  belongs to  $\mathcal{C}$ .

The first axiom basically tells us that the class  $\mathcal{C}$  is not empty and the names of the other three axioms are explicit enough. A word of caution is however necessary about the “2 out of 3” axiom, because, as the reader will certainly have noticed, two thirds of the usual kind of a “2 out of 3” axiom are missing. The first reason is that the full axiom would be too strong, as illustrated by the following example

$$\begin{array}{ccccc} \emptyset & \longrightarrow & \Delta[0] & \xrightarrow{d^0} & S^0 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta[0] & \xrightarrow{d^0} & S^0 & \xrightarrow{id} & S^0 \end{array}$$

in which the left square and the outer square are (homotopy) pull-backs, but the right square is not. The second reason is that the remaining third of the axiom (composition of morphisms in  $\mathcal{C}$ ) is implicit in the definition:

**Proposition 1.2.** *Let  $\mathcal{C}$  be a distinguished class. If  $\phi : f \rightarrow g$  and  $\psi : g \rightarrow h$  belong to  $\mathcal{C}$ , then so does  $\psi\phi$ .*

*Proof.* The homotopy colimit of the following diagram  $F$  in *Arrows*:

$$f \xrightarrow{\phi} g \xleftarrow{id_g} g \xrightarrow{\psi} h$$

is homotopy equivalent to  $h$ . Moreover the morphism  $f \rightarrow \text{hocolim} F$  can be identified with the composition  $\psi\phi$ . Since all morphisms in this diagram belong to  $\mathcal{C}$ , by condition (4) in Definition 1.1 so does  $\psi\phi$ .  $\square$

Where do these axioms come from? Consider an arbitrary distinguished class  $\mathcal{C}$  and a homotopy pull-back square  $f \rightarrow g$ . By (1), weak equivalences belong to  $\mathcal{C}$ , so that by homotopy invariance – axiom (2) – we are allowed to replace the original square by a pull-back square  $f \rightarrow g$  in which both  $f$  and  $g$  are fibrations.

In order to simplify the argument, let us suppose that the map  $f$  is of the form  $X \rightarrow *$ , i.e.  $X$  is the (homotopy) fiber of  $g$ . We have seen that any map yields a diagram of preimages of simplices and we have here morphisms  $dg(\sigma) \rightarrow \Delta[n]$  for any simplex  $\sigma \in B$ , forming a natural transformation of diagrams indexed by  $\mathbf{B}$ .

Since  $g$  is a fibration, the commutative squares

$$\begin{array}{ccc} dg(\sigma) & \xrightarrow{\cong} & dg(\tau) \\ \downarrow & & \downarrow \\ \Delta[n] & \xrightarrow{\cong} & \Delta[m] \end{array}$$

are weak equivalences and thus belong to  $\mathcal{C}$ . The Puppe property (4) implies therefore that so does any square

$$\begin{array}{ccc} dg(\sigma) & \longrightarrow & \text{hocolim}_{\mathbf{B}} dg \simeq A \\ \downarrow & & \downarrow g \\ \Delta[n] & \longrightarrow & \text{hocolim}_{\mathbf{B}} \Delta[m] \simeq B \end{array}$$

This explains why homotopy pull-back squares belong to *any* distinguished class. It is obvious that homotopy pull-back squares satisfy the first three conditions of Definition 1.1, but to prove that they actually satisfy the Puppe property is somewhat more involved. One starts with push-out diagrams and goes on by induction on gluing cells (for the diagrams can be assumed to be indexed by simplex diagrams), see [2, Theorem 6.2] for more details.

**Theorem 1.3.** *The collection of homotopy pull-backs is the smallest distinguished class of morphisms in  $Arrows$ .*  $\square$

The “2 out of 3” axiom, (3) in Definition 1.1, is not used in the proof of Theorem 1.3. It is however essential to understand a distinguished class in terms of transition functions. In the category of  $Arrows$  a homotopy pull-back  $\sigma \rightarrow f$  for which the range of  $\sigma$  is contractible, is called a *homotopy fiber* of  $f$ .

**Corollary 1.4.** *Let  $\mathcal{C}$  be a distinguished class. Then  $\phi : f \rightarrow g$  belongs to  $\mathcal{C}$  if and only if the morphism  $\pi : \sigma \rightarrow \tau$  does so for any commutative square of the form:*

$$\begin{array}{ccc} \sigma & \xrightarrow{\pi} & \tau \\ \downarrow & & \downarrow \\ f & \xrightarrow{\phi} & g \end{array}$$

where  $\sigma \rightarrow f$  and  $\tau \rightarrow g$  are respectively homotopy fibers of  $f$  and  $g$ .  $\square$

## 2. FIBERWISE LOCALIZATION

Let  $\phi$  be a morphism in  $Arrows$ , i.e. a commutative square. Recall that a map of spaces  $f$  is called  $\phi$ -local if the map of spaces  $\text{map}(\phi, f)$  is a weak equivalence.

According to [1] (see also [4] and [5]), there is a functor  $L_\phi : Arrows \rightarrow Arrows$  and a natural transformation  $f \rightarrow L_\phi f$  (called localization) such that:

- $L_\phi f$  is  $\phi$ -local;
- the map of spaces  $\text{map}(L_\phi f, g) \rightarrow \text{map}(f, g)$  is a weak equivalence, for any  $\phi$ -local  $g$ .

In general it is very difficult to understand the localization with respect to an arbitrary morphism  $\phi$ , or even the local objects. Let us concentrate on the situation when  $\phi$  is a morphism of the form

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ \star & \longrightarrow & \star \end{array}$$

in which case we will denote  $L_\phi$  by  $L_u$ . We warn the reader that it is extremely important to fix a convention and decide once and for all which arrows are represented vertically and which horizontally, since the situation is not symmetric. For

us the vertical arrows are the objects in the category *Arrows*, and the horizontal ones are the morphisms. Localization with respect to the square  $u \rightarrow *$  would be  $u$ -nullification, a way more complicated functor than the one we study.

**Proposition 2.1.** *A map  $f$  is  $L_u$ -local in *Arrows* if and only if, any homotopy fiber  $\text{Fib}(f)$  is  $L_u$ -local in *Spaces*, i.e. the map of spaces  $\text{map}(u, \text{Fib}(f))$  is a weak equivalence.  $\square$*

We identify then the localization  $L_u$  in *Arrows* with fiberwise localization in *Spaces*, a construction which is well explained in [4, Theorem 1.F.1].

**Theorem 2.2.** *Let  $f : X \rightarrow Y$  be a map of spaces. Then  $L_u f$  is weakly equivalent to a morphism  $g : E \rightarrow Y$  and the localization morphism  $f \rightarrow L_u f$  is weakly equivalent to a commutative square of the form  $\psi : f \rightarrow g$  where*

- (1) *the map  $\psi_0 : X \rightarrow E$  is an  $L_u$ -equivalence,*
- (2) *the map  $\psi_1 : Y \rightarrow Y$  is the identity,*
- (3) *for any commutative square  $\sigma \xrightarrow{\pi} \tau$  in which  $\sigma \rightarrow f$  and  $\tau \rightarrow g$  are ho-*

$$\begin{array}{ccc} \sigma & \xrightarrow{\pi} & \tau \\ \downarrow & & \downarrow \\ f & \xrightarrow{\psi} & g \end{array}$$

*motopy fibers,  $\pi$  induces an  $L_u$ -equivalence in *Spaces* between the domains  $\text{Fib}(f) \rightarrow \text{Fib}(g)$ .  $\square$*

### 3. HOMOTOPY PULL-BACKS SQUARES UP TO LOCALIZATION

In Theorem 1.3 we saw that any distinguished class containing all weak equivalences must also contain all homotopy pull-backs. The analogous statement for  $L_\phi$ -equivalences should involve  $L_\phi$ -homotopy pull-backs.

**Definition 3.1.** Given a morphism  $\psi : f \rightarrow g$  in *Arrows*, consider the homotopy pull-back  $P = \text{holim}_Y(Y \xrightarrow{\psi_1} V \xleftarrow{g} U)$  and denote by  $h$  the map  $P \rightarrow Y$ . Then  $\psi$  is called an  $L_\phi$ -homotopy pull-back if the morphism  $f \rightarrow h$  is an  $L_\phi$ -equivalence.

There is a more amenable description of  $L_\phi$ -homotopy pull-backs when  $\phi$  is of the form studied in the previous section.

**Proposition 3.2.** *If  $u$  is a map of spaces, then  $\psi : f \rightarrow g$  is an  $L_u$ -homotopy pull-back if and only if  $L_u \psi$  is a homotopy pull-back.  $\square$*

Using this characterization, we can now prove our main result by a reduction to the case of homotopy pull-backs, Theorem 1.3.

**Theorem 3.3.** *Let  $u$  be a map of spaces. The collection of  $L_u$ -homotopy pull-backs is the smallest distinguished class containing all  $L_u$ -equivalences.  $\square$*

To illustrate this theorem we offer an application with a classical flavor.

**Corollary 3.4.** *Let  $f : X \rightarrow Y$  be a map over a connected space  $Y$  and denote by  $F$  its homotopy fiber. Assume that any morphism  $\sigma \rightarrow \tau$  in  $\mathbf{Y}$  induces an  $L_u$ -homotopy pull-back  $df(\sigma) \rightarrow df(\tau)$ . Then the map  $df(\sigma) \rightarrow F$  is an  $L_u$ -equivalence for any simplex  $\sigma$  in  $Y$ .  $\square$*

## 4. CONCLUDING REMARKS

Theorem 3.3 yields a “Puppe Theorem” for any localization functor, since property (4) in Definition 1.1 holds for the class of  $L_u$ -homotopy pull-back squares. Explicitly, if  $F \rightarrow G$  is a natural transformation of diagrams indexed by some small category  $I$  inducing  $L_u$ -equivalences on homotopy fibers, then one also knows the homotopy fiber of the map  $\text{hocolim}_I F \rightarrow \text{hocolim}_I G$ , at least up to  $L_u$ -localization. Such a statement was suggested to us by Emmanuel Dror Farjoun when we explained him our previous work [7].

Let us end with an open question. In all the examples we have seen of distinguished classes, the transition functions are  $L_u$ -equivalences for some map  $u$ . One could go the other way around and start with a certain class of “transition functions”  $\mathcal{D}$  and define  $\mathcal{D}$ -homotopy pull-backs, as in Corollary 1.4, to be the collection  $\mathcal{C}(\mathcal{D})$  of morphisms  $\phi : f \rightarrow g$  in *Arrows* for which the maps induced on homotopy fibers of  $f$  and  $g$  belong to  $\mathcal{D}$ .

**Question 4.1.** Which are the axioms  $\mathcal{D}$  must satisfy so that  $\mathcal{C}(\mathcal{D})$  is a distinguished class? And are there other distinguished classes than those of  $L_u$ -homotopy pull-backs?

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# STABLE MAPPING CLASS GROUPS OF 4-MANIFOLDS AND COBORDISM CATEGORIES

JEFFREY GIANIRACUSA

ABSTRACT. In the first half of this talk I discuss my recent work with stable mapping class groups of simply-connected smooth 4-manifolds. Defining stabilization by taking connected sums with  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ , the stable mapping class group turns out to be isomorphic to the stable group of automorphisms of the intersection form. The proof requires an analysis of Dehn twists around boundary 3-spheres. We find that if a manifold contains  $\mathbb{C}P^2$  as a connected summand then Dehn twists around boundary 3-spheres are isotopically trivial.

In the second part of the talk I discuss cobordism categories. One may construct a cobordism category of 4-manifolds bounded by 3-manifolds and then enrich this structure with (isotopy classes of) diffeomorphisms. Morava calls this a gravity category, and according to Morava and Witten a theory of topological gravity should be a sort of representation of this object. I will describe how the classifying space of the gravity category of simply-connected 4-manifolds bounded by ordinary 3-spheres is naturally an infinite loop space and is in fact equivalent to the Hermitian K-theory of the integers. A similar result holds when one restricts to spin 4-manifolds.

## 1. THE STABLE MAPPING CLASS GROUP

Let  $M$  be a compact oriented smooth 4-manifold (possibly with boundary); we further assume that  $M$  is connected and simply-connected. Associated to  $M$  is a *mapping class group*

$$\Gamma(M) := \pi_0 \text{Diff}(M, \partial M);$$

i.e. the group of isotopy classes of diffeomorphisms of  $M$  which restrict to the identity on the boundary. This is an object of intrinsic interest to geometers and topologists, and while mapping class groups of surfaces have been extensively studied, we know comparatively little about these groups in dimension 4. However, I would like to make the point that we already have many of the tools necessary to understand the mapping class group stably.

Our strategy shall be to compare the mapping class group to a more algebraic object: the group  $\text{Aut}(M)$  of automorphisms of the intersection form on  $H_2(M, \mathbb{Z})$ . There is a natural map  $\Gamma(M) \rightarrow \text{Aut}(M)$  given by sending an isotopy class to the induced automorphism of  $H_2$ .

Let us begin by briefly recalling some facts about this map. For surfaces, the analogous map  $\Gamma(F_g) \rightarrow Sp_{2g}(\mathbb{Z})$  pulls back only the odd half of the Miller-Morita-Mumford characteristic classes in cohomology, so at the cohomological level the two groups are neither too similar nor too different. However, the situation is markedly simpler for *topological* 4-manifolds.

**Theorem 1** ([Qui86]). *For a simply-connected topological 4-manifold  $M$ ,  $\pi_0 \text{Homeo}(M) \cong \text{Aut}(M)$ .*

Of course, once we set foot in the land of smooth manifolds, the situation becomes somewhat disconcerting, for we have

**Theorem 2** ([Rub99]). *The homomorphism  $\pi_0 \text{Diff}(M) = \Gamma(M) \rightarrow \text{Aut}(M)$  can have non-finitely generated kernel!*

Ruberman's theorem was proven using gauge theoretic invariants, but gauge theory tends to only detect properties which are *unstable* with respect to connected sum. For instance, the Donaldson polynomial vanishes after a single connected sum with  $\mathbb{C}P^2$ . Furthermore, if we allow ourselves to start taking connected sums with  $S^2 \times S^2$  then a well-known theorem of Wall [Wal64b] tells us that the stable diffeomorphism type is determined entirely by the intersection form. Motivated by these examples of how the unusual phenomena of smoothness in dimension 4 tend to go away in stabilization, there is hope that we may understand  $\Gamma(M)$  *stably*.

The particular stabilization I want to consider is taking connected sums with  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ . Though perhaps less familiar than using  $S^2 \times S^2$ , this stabilization makes sense for the following two reasons. (i) There is a diffeomorphism  $(S^2 \times S^2) \# \overline{\mathbb{C}P^2} \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ , so our stabilization process automatically implicitly contains the more familiar stabilization with respect to  $S^2 \times S^2$ . (ii) Unimodular forms are either *even* or *odd*, and *definite* or *indefinite*. Since connected sum of manifolds corresponds to direct sum of intersection forms, the most stable case is that of an odd indefinite form, and such a form is isomorphic to  $n(1) \oplus m(-1)$ . The intersection form of  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  is  $(1) \oplus (-1)$ , so our stabilization process puts us immediately into the land of odd indefinite forms and sends the number of  $(1)$ 's and  $(-1)$ 's both to infinity.

To stabilize the mapping class group, one must have a way to extend a diffeomorphism across a connected sum. In general this is impossible since one needs a fixed disc in which to perform the cutting and pasting, and there might not be a fixed disc anywhere. Up to a non-canonical isotopy one does have a fixed disc, but then the result may depend on the choice of isotopy. Instead, we shall use manifolds with boundary and stabilize by gluing along the boundary. Let  $M$  be bounded by some number of ordinary 3-spheres, and let  $X$  denote  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  with the interiors of two discs removed. We may glue  $X$  along a selected boundary component of  $M$  and then iterate by gluing along the remaining boundary of  $X$ . Extension by the identity on  $X$  determines a system of maps

$$\Gamma(M) \rightarrow \Gamma(MX) \rightarrow \Gamma(MX^2) \rightarrow \dots \rightarrow \operatorname{colim}_n \Gamma(MX^n) := \Gamma_\infty(M)$$

with colimit defined to be the *stable mapping class group* of  $M$ . Similarly, we define the *stable automorphism group* of  $M$ ,  $\operatorname{Aut}_\infty(M)$ . Note that gluing along a boundary 3-sphere induces block addition of intersection forms.

**Theorem 3** ([Gia05]). *For smooth compact oriented simply-connected 4-manifolds  $M$  bounded by a collection of ordinary 3-spheres,  $\Gamma_\infty(M)$  is independent of  $M$ , and in fact*

$$\Gamma_\infty(M) \cong \operatorname{Aut}_\infty(M) \cong O_{\infty, \infty}(\mathbb{Z}).$$

Here  $O_{\infty, \infty}(\mathbb{Z})$  is the group of automorphisms of the quadratic form  $\infty(1) \oplus \infty(-1)$  on  $\mathbb{Z}^\infty$ . This group is closely related to the Hermitian  $K$ -theory of the integers, which has been studied for example in [BK05].

The theorem essentially follows from theorems due to Wall, Kreck, and Quinn which relate the isotopy and pseudo-isotopy of  $M$  to the automorphism group of  $M$ . (Two diffeomorphisms are pseudo-isotopic if they are the restrictions to  $M \times \{0\}$  and  $M \times \{1\}$  of a diffeomorphism of  $M \times I$ .)

**Theorem 4** ([Wal64a]). *If  $M$  is of the form  $N \# (S^2 \times S^2)$  with  $N$  indefinite, then  $\Gamma(M) \rightarrow \operatorname{Aut}(M)$  is surjective.*

**Theorem 5** ([Kre79]). *On pseudo-isotopy we always have an injection  $\Gamma(M)/(pseudo\text{-isotopy}) \hookrightarrow \operatorname{Aut}(M)$ .*

**Theorem 6** ([Qui86]). *If  $\phi \in \operatorname{Diff}(M)$  is pseudo-isotopic to the identity, then for  $k$  large enough its extension by identity on  $S^2 \times S^2$  to  $\Phi \in \operatorname{Diff}(M \# k(S^2 \times S^2))$  is isotopic to the identity.*

Taken together, these theorems imply

**Proposition 7.** *For  $k$  large enough there is a natural inclusion  $\operatorname{Aut}(M) \hookrightarrow \Gamma(M \# k(S^2 \times S^2))$ . (Of course, this also holds with  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  instead of  $S^2 \times S^2$ .)*

We would now like to use this inclusion to construct an inverse  $\operatorname{Aut}_\infty(M) \dashrightarrow \Gamma_\infty(M)$ , but a difficulty arises: Kreck's theorem on injectivity refers to closed manifolds (or at least, diffeomorphisms which do not necessarily fix the boundary), we however need a boundary component fixed under the diffeomorphisms in order to stabilize.

The resolution of this difficulty comes from an analysis of the homomorphism induced by closing up the manifold with discs. Let  $\widetilde{M}$  denote the result of sewing a disc onto each boundary component of  $M$ ; it is not hard to see that  $\Gamma(M) \rightarrow \Gamma(\widetilde{M})$  is surjective.

**Proposition 8.** *If  $M$  contains  $\mathbb{C}P^2$  as a connected summand then the homomorphism  $\Gamma(M) \rightarrow \Gamma(\widetilde{M})$  is an isomorphism.*

*Proof.* If  $M$  is bounded by  $n$  3-spheres, then there is a fibration

$$\operatorname{Diff}(M, \partial M) \longrightarrow \operatorname{Diff}(\widetilde{M}) \xrightarrow{ev} \operatorname{Emb}(n \text{ discs}, \widetilde{M})$$

and retraction of the disc to its center gives a homotopy equivalence between the embedding space and the framed configuration space  $FC(n, \widetilde{M})$  of  $n$  (ordered) framed points in  $\widetilde{M}$ . This space is connected and has fundamental group  $(\mathbb{Z}/2)^n$  with generators corresponding to rotations of each of the framings (there is no contribution from the topology of  $M$  since  $M$  is simply connected). From the long exact homotopy sequence we then see that  $\Gamma(M) \rightarrow \Gamma(\widetilde{M})$  is surjective with kernel consisting of precisely the Dehn twists<sup>1</sup> around the boundary components, though some of these might already be zero in  $\Gamma(M)$ , depending on how large the image of  $ev_*$  is in  $\pi_1 FC(n, \widetilde{M})$ .

For the moment, let us specialize to  $\mathbb{C}P^2 -$  (two discs). We will show that  $ev_* : \pi_1 \text{Diff}(\mathbb{C}P^2) \rightarrow \pi_1 FC(2, \mathbb{C}P^2)$  is surjective, and hence the Dehn twist around each of the two boundary components is isotopic to the identity even before gluing on the discs.

The circle action  $\lambda \cdot [z_1, z_2, z_3] \mapsto [\lambda z_1, z_2, z_3]$  contains spaces  $p = [1, 0, 0]$  and  $q = [0, 0, 1]$  in its fixed point set and the  $S^1$  representations on the tangent spaces at  $p$  and  $q$  are  $1 \oplus 0$  and  $1 \oplus 1$  respectively. Therefore the generator of  $\pi_1 S^1$  is sent to the element of  $\pi_1 FC(2, \mathbb{C}P^2) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$  corresponding to the nontrivial rotation of the framing at  $p$  and the trivial rotation at  $q$ . Using a second circle action we see that the nontrivial rotation at  $q$  is also in the image, so  $ev_*$  is surjective on  $\pi_1$  and thus the Dehn twists in  $\mathbb{C}P^2 -$  (2 discs) are both isotopic to the identity.

Returning now to the case of  $M' \# \mathbb{C}P^2$ , we may assume that one of the boundary components lies in the  $\mathbb{C}P^2$  summand. The above analysis of  $\mathbb{C}P^2$  implies that the twist around this component is isotopic to the identity. But then the twist around any boundary component is isotopic to the identity, for the boundary components are indistinguishable up to diffeomorphism.  $\square$

To summarize what we now know about the stabilization process:

- After a single step of stabilization  $\Gamma(M)$  becomes independent of the number of boundary 3-spheres;
- In the colimit,  $\Gamma_\infty(M) \cong \text{Aut}_\infty(M) \cong O_{\infty, \infty}(\mathbb{Z})$  and is independent of  $M$  entirely.

## 2. TOPOLOGICAL GRAVITY AND COBORDISM CATEGORIES

A cobordism category  $\mathcal{C}$  is roughly a category where the objects are manifolds and the morphisms are cobordisms, with perhaps some additional data attached to everything in sight. Disjoint union provides a symmetric monoidal structure on  $\mathcal{C}$ . Following Atiyah and Segal, a quantum field theory is a sort of representation of a cobordism category; that is, a monoidal functor

$$F : (\mathcal{C}, \sqcup) \rightarrow (\mathbf{Vect}, \otimes),$$

where  $\mathbf{Vect}$  is some appropriately linear and monoidal category, such as Hilbert spaces.

A symmetric monoidal product on a category gives rise to an infinite loop space structure on the nerve of the category. The nerve of  $\mathbf{Vect}$  will be some sort of  $K$ -theory infinite loop space. A monoidal functor  $F : \mathcal{C} \rightarrow \mathbf{Vect}$  will thus produce an element in the (appropriate version of the)  $K$ -theory of  $|\mathcal{C}|$ . Therefore, a very rough first step towards constructing or classifying field theories might be to try to understand the homotopy type of  $|\mathcal{C}|$  and some flavor of its  $K$ -theory.

Witten and Morava have proposed that a theory of *topological quantum gravity* in 3+1 dimensions should be a representation of a cobordism 2-category of 4-manifolds, where the 2-morphisms are diffeomorphisms. If one constructs such a category using *all* 3-manifolds and *all* 4-manifolds, then the homotopy type of this category is well-understood by the work of Galatius, Madsen, Tillmann, and Weiss [TMWG05]. However, their theorem only applies once we have thrown everything into the pot; it tells us very little about variations on the category.

Morava [Mor04] has pointed out that restricting to spin 4-manifolds bounded by collections of ordinary 3-spheres yields an interesting connection with  $SU(2)$ -equivariant Tate cohomology. As a first step towards understanding Morava's category let us consider a slightly easier category, namely simply-connected 4-manifolds bounded by 3-spheres (one must be careful that compositions won't introduce nontrivial  $\pi_1$ ) and 2-morphisms given by *isotopy classes* of diffeomorphisms.

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<sup>1</sup>In this setting, a Dehn twist is given by a 3-sphere embedded with trivial normal bundle together with loop  $\alpha \in \Omega \text{Diff}(S^3)$ . One constructs a diffeomorphism of  $M$  by defining it to be the identity outside of a tubular neighborhood of the sphere, and traversing  $\alpha$  across the levels of the tubular neighborhood. Note that  $\pi_1 \text{Diff}(S^3) \cong \mathbb{Z}/2$ , so all Dehn twists are elements of order 2.

Tillmann's generalized group completion argument [Til97] together with our understanding of the stable mapping class group now give us:

**Theorem 9** ([Gia05]). *Let  $\mathcal{C}$  be the (topological) 2-category consisting of ordinary 3-spheres, simply connected 4-manifolds, and 2-morphisms given by isotopy classes of diffeomorphisms. Then*

$$\Omega|\mathcal{C}| \simeq L_{HZ}(\mathbb{Z}^2 \times BO_{\infty, \infty}(\mathbb{Z})),$$

where  $L_{HZ}$  is Bousfield localization at ordinary integral homology.

For simply-connected spin manifolds, a similar result holds away from the prime 2; the spin condition implies that there can be no  $\mathbb{C}P^2$  with which to kill Dehn twists around boundary components, but fortunately these twists are all order 2. This time the category gives essentially the higher algebraic  $K$ -theory of even unimodular lattices over  $\mathbb{Z}$ .

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# Vector bundles on Orbispaces

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Orbispaces are spaces with extra structure. The main examples come from compact Lie group actions  $X \curvearrowright G$  and are denoted  $[X/G]$ , their underlying space being  $X/G$ . By definition, every orbispace is locally of the form  $[X/G]$ , but the group  $G$  might vary.

To be more precise, an orbispace is a topological stack which is locally equivalent to  $[X/G]$  for  $G$  a Lie group and  $X$  a  $G$ -CW-complex [2] [4] [5]. However, if all the stabilizer groups are finite, there exists a more concrete alternative definition [1].

**Definition 1** *An orbispace is a map  $p : E \rightarrow Y$  satisfying the following condition: there exists an open cover  $\{U_i\}$  of  $Y$  and finite group actions  $X_i \curvearrowright G_i$  such that  $p^{-1}(U_i) \rightarrow U_i$  is fiberwise homotopy equivalent to  $(X_i \times EG_i)/G_i \rightarrow X_i/G_i$ . The space  $Y$  is called the coarse moduli space, and the space  $E$  is called the total space.*

*A morphism of orbispaces  $(E, Y) \rightarrow (E', Y')$  is a commutative diagram:*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$

*If  $g = g'$ , there may exist 2-morphisms  $(E, Y) \mathfrak{B} (E', Y')$  between morphisms  $(f, g)$  and  $(f', g')$ . A 2-morphism is a homotopy  $h : E \times [0, 1] \rightarrow E'$  such that  $p' \circ h_t = g \circ p$  for all  $t \in [0, 1]$ :*

$$\begin{array}{ccc} & f & \\ E & \xrightarrow{\quad} & E' \\ & \Downarrow h & \\ & f' & \\ Y & \xrightarrow{g} & Y' \end{array} \quad (1)$$

*If  $g \neq g'$ , there are no 2-morphisms. Two 2-morphisms are considered to be the same if they are homotopic to each other relatively to the endpoints.*

We have two notions of point: those of the total space, and those of the coarse moduli space. If the stabilizer groups are not finite, Definition 1 is no

longer valid, but these two notions still exist. The points “of the total space” form the objects of a groupoid, and the points of the coarse moduli space are the isomorphism classes of objects. Given an object of this groupoid, its automorphism group corresponds to the stabilizer of the action.

An orbispace always comes with a map to its coarse moduli space. By a suborbispace  $\mathfrak{X}' \subset \mathfrak{X}$ , we shall mean an orbispace obtained by pulling back along a subspace of the coarse moduli space.

If  $\mathfrak{X} = (E \rightarrow Y)$  is an orbispace as in Definition 1, a vector bundle  $V$  over  $\mathfrak{X}$  is a vector bundle  $V \rightarrow E$  equipped with a flat connection in the direction of the fibers of  $p$ . Otherwise, one needs to use the classical notion of vector bundles on stacks.

It is tempting to define  $K$ -theory as the Grothendieck group of vector bundles. But, as is shown in [3], this is not always a good idea. For example, one needs the following technical property in order to prove the excision axiom:

**Definition 2** *An orbispace  $\mathfrak{X}$  has enough vector bundles if for every suborbispace  $\mathfrak{X}' \subset \mathfrak{X}$  and every (finite dimensional) vector bundle  $V$  on  $\mathfrak{X}'$ , there exists a vector bundle  $W$  on  $\mathfrak{X}$  and a linear embedding  $V \hookrightarrow W$ .*

This condition is not always satisfied (Example 4, see also section 5 of [3]), but it is if  $\mathfrak{X}$  is compact with finite stabilizer (Corollary 6). We first establish the following connection with global quotients:

**Theorem 3** *Let  $\mathfrak{X}$  be a compact orbispace (i.e. its coarse moduli space is compact). Then the following are equivalent:*

1.  $\mathfrak{X}$  is a global quotient by a compact Lie group i.e.  $\mathfrak{X} = [X/G]$  for some compact Lie group  $G$  acting on a compact space  $X$ .
2.  $\mathfrak{X}$  has enough vector bundles.
3. There exists a vector bundle  $W$  on  $\mathfrak{X}$  such that for every point  $x$  (“of the total space”) the action of  $\text{Aut}(x)$  on  $W_x$  is faithful.

*Proof.* 1.  $\Rightarrow$  2. Let  $X \curvearrowright G$  be such that  $\mathfrak{X} \simeq [X/G]$ , and let  $X' \subset X$  be the  $G$ -invariant subspace corresponding to  $\mathfrak{X}' \subset \mathfrak{X}$ . Let  $V$  be a vector bundle on  $\mathfrak{X}'$  and let  $\tilde{V}$  be the corresponding  $G$ -equivariant vector bundle on  $X'$ . It is well known that any equivariant vector bundle  $\tilde{V}$  on a compact space  $X'$  embeds in one of the form  $X' \times M$ , where  $M$  is a representation of  $G$ . Let  $W$  be the vector bundle on  $\mathfrak{X}$  corresponding to  $X \times M \rightarrow X$ . Since  $\tilde{V}$  embeds in  $X \times M$ , the bundle  $V$  embeds in  $W$ .

2.  $\Rightarrow$  3. Suppose that  $\mathfrak{X}$  has enough vector bundles, and let  $\{U_i\}$  be a finite cover of  $\mathfrak{X}$  such that  $U_i \simeq [X_i/G_i]$ . Let  $M_i$  be faithful representations of  $G_i$ , and let  $V_i$  be the vector bundles on  $U_i$  corresponding to  $X_i \times M_i \rightarrow X_i$ . Since  $M_i$  is faithful, the stabilizer groups act faithfully on the fibers of  $V_i$ . Let  $W_i$  be vector bundles on  $\mathfrak{X}$  such that  $V_i \hookrightarrow W_i$ , and let  $W := \bigoplus W_i$ . Clearly, the stabilizer groups act faithfully on the fibers of  $W$ .

3.  $\Rightarrow$  1. Let  $P$  be the total space of the frame bundle of  $W$ . The stabilizer groups act faithfully on the fibers of  $W$ , hence they act freely on the fibers of  $P$ . Having no stabilizer groups,  $P$  is a space (as opposed to an orbispace). We have  $\mathfrak{X} = [P/O(n)]$  and so  $\mathfrak{X}$  is a global quotient.  $\square$

Note that there exist compact orbispaces which are not global quotients.

**Example 4** Let  $P \rightarrow S^3$  be the principal  $BS^1$ -bundle classified by

$$1 \in [S^3, B(BS^1)] = \mathbb{Z}.$$

Then  $\mathfrak{X} := [P/ES^1]$  is not a global quotient by a Lie group.

Indeed, suppose that  $\mathfrak{X} = [X/G]$ . Since  $\mathfrak{X} \rightarrow S^3$  is homotopically non-trivial, the map  $X \rightarrow S^3$  needs to be a non-trivial  $G$ -fiber bundle with fiber  $S^1 \backslash G \cong G$ . Let  $H = \text{Aut}_G(S^1 \backslash G)$  be the structure group of that bundle. All compact Lie groups have trivial  $\pi_2$ , therefore  $[S^3, BH] = \pi_3 BH = \pi_2 H = 0$ . The bundle  $X \rightarrow S^3$  is trivial, a contradiction.

The above example shows that more orbispaces are global quotients by topological groups, than by compact Lie groups. Actually, all orbispaces are global quotients by topological groups.

The following theorems should have been the main results of [1]. Unfortunately the proofs have some gaps, but we believe that we are now able to fill them.

**Almost-Theorems 5** 1. There exists a topological group  $G_1$  such that every orbispace is a global quotient by  $G_1$ .

2. There exists an ind-Lie group  $G_2$  such that every orbispace with finite stabilizers is a global quotient by  $G_2$ .

3. Every compact orbispace with finite stabilizers is the global quotient by some compact Lie group.

The group  $G_1$  can be any group with the following property. First, it must contain all compact Lie groups as subgroups. Moreover, if  $K, K' \subset G_1$  are compact Lie subgroups, and if  $f : K \rightarrow K'$  is a monomorphism, then the space

$$\{g \in G_1 \mid \text{Ad}(g)|_K = f\}$$

must be contractible. In particular, the group  $G_1$  itself has to be contractible.

The group  $G_2$  be described more explicitly. It is the colimit of  $U(n!)$  under the maps  $U(n!) \rightarrow U((n+1)!) : A \mapsto A \otimes \mathbf{1}_{n+1}$ .

**Corollary 6** If  $\mathfrak{X}$  is a compact orbispace with finite stabilizer groups, then  $\mathfrak{X}$  has enough vector bundles.

*Sketch a proof of Theorems 5.* Theorem 5.3 is an easy corollary of Theorem 5.2, so we concentrate on the proofs of Theorems 5.1 and 5.2.

The coarse moduli space of an orbispace is stratified by the type of stabilizer group. Let  $P$  be the poset of isomorphism classes of groups (compact Lie groups

for the proof of Theorem 5.1, finite groups for the proof of Theorem 5.2). For now on, all spaces will be stratified by  $P$ . Similarly, maps between stratified spaces will always send a stratum of the source to the corresponding stratum of the target.

Let an *orbispace structure* on a stratified space  $Y$  be an orbispace  $\mathfrak{X}$  over  $Y$ , inducing the given stratification. One can then build a universal orbispace  $\underline{\mathbf{Orb}}$  with coarse moduli space  $\underline{\mathbf{Orb}}$ , such that for each stratified space  $Y$  we then have a bijection

$$\left\{ \begin{array}{l} \text{Homotopy classes of} \\ \text{stratified maps } Y \rightarrow \underline{\mathbf{Orb}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{orbispace structures on } Y \end{array} \right\}.$$

Let  $G (= G_1 \text{ or } G_2)$  be a topological group and  $Y$  a stratified space. A *G-quotient structure* on  $Y$  is a  $G$ -space  $X$  over  $Y$  inducing a stratified homeomorphism  $X/G \simeq Y$ , where the stratification on  $X/G$  is by stabilizer groups. Again, there exists a universal  $G$ -space  $\underline{EG}$  with quotient  $\underline{BG} = \underline{EG}/G$ , such that for every stratified space  $Y$  we have a bijection

$$\left\{ \begin{array}{l} \text{Homotopy classes of} \\ \text{stratified maps } Y \rightarrow \underline{BG} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ G\text{-quotient structures on } Y \end{array} \right\}.$$

The functor  $(X \mathfrak{H} G) \mapsto [X/G]$  is represented by a stratified map  $\underline{BG} \rightarrow \underline{\mathbf{Orb}}$ . The theorem is then equivalent to the existence of a section

$$\underline{BG} \xrightarrow{\quad \dashrightarrow \quad} \underline{\mathbf{Orb}}.$$

The obstructions to the existence of such a section lie in  $H^{k+1}(\underline{\mathbf{Orb}}; \pi_k(\text{Fiber}))$ , which can be computed to be zero [1].

Note that the homotopy type of the fiber varies with the stratum, and so  $\pi_k(\text{Fiber})$  is not a local system but rather a constructible sheaf. The argument therefore needs a new type of obstruction theory. This is where the proof in [1] has gaps.

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# THE RATIONAL HOMOTOPY TYPE OF SPACES OF LONG KNOTS IN CODIMENSION $> 2$

P. LAMBRECHTS

(JOINT WORK WITH G. ARONE, V. TURCHIN AND I. VOLIĆ)

ABSTRACT. We explain that the Vassiliev spectral sequence computing the rational homology of the space of long knots in codimension  $> 2$  collapses at  $E_1$ . The proof is a combination of Goodwillie approach to embedding spaces and of Kontsevich formality theorem. This result completely determines the rational homotopy type of that space of long knots.

We also state a certain coformality result which implies the collapsing of a Bousfield-Kan spectral sequence computing the rational homotopy groups of the space of long knots.

## The space of long knots.

Let  $d \geq 3$  and fix the linear embedding

$$\epsilon: \mathbf{R} \rightarrow \mathbf{R}^d, t \mapsto (t, 0, \dots, 0).$$

A *long knot* is a smooth embedding  $f: \mathbf{R} \hookrightarrow \mathbf{R}^d$  such that  $f$  agrees with  $\epsilon$  outside of a compact set. We denote by  $\mathcal{K}'^d = \text{Emb}(\mathbf{R}, \mathbf{R}^d; \epsilon)$  the space of long knots in  $\mathbf{R}^d$ . This space is closely related to the usual space of knots  $\text{Emb}(S^1, S^d)$ . Indeed it is easy to check that  $\mathcal{K}'^d$  is homotopy equivalent to the fibre of the map

$$\text{Emb}(S^1, S^d) \rightarrow V_2(\mathbf{R}^{d+1}) = SO(d+1)/SO(d-1), g \mapsto (g(1), g'(1)/\|g'(1)\|)$$

where  $V_2(\mathbf{R}^{d+1})$  is the Stiefel manifold of orthogonal 2-frames in  $\mathbf{R}^{d+1}$ .

To simplify the exposition we will study here a slight variation of the space  $\mathcal{K}'^d$  that we define now. Consider the space  $\text{Imm}(\mathbf{R}, \mathbf{R}^d; \epsilon)$  of *immersions*  $f: \mathbf{R} \rightarrow \mathbf{R}^d$  that agree with  $\epsilon$  at infinity. There is an obvious inclusion

$$(1) \quad \text{Emb}(\mathbf{R}, \mathbf{R}^d; \epsilon) \hookrightarrow \text{Imm}(\mathbf{R}, \mathbf{R}^d; \epsilon).$$

We denote by  $\mathcal{K}^d$  the homotopy fibre of that map and this is the space that we will study in this report. Notice that by a classical result of Smale,  $\text{Imm}(\mathbf{R}, \mathbf{R}^d; \epsilon) \simeq \Omega S^{d-1}$ . Moreover the map (1) turns out to be homotopically trivial ([11, Proposition 5.17]). Therefore we have a homotopy equivalence

$$\mathcal{K}^d \simeq \mathcal{K}'^d \times \Omega^2 S^{d-1}.$$

## The Vassiliev approach.

In the late 1980's Vassiliev [13] invented his method of “discriminant” to study the homology of spaces of maps without singularities. Spaces of long knots are a special case of this and Vassiliev's approach gives a spectral sequence converging (at least when  $d \geq 4$ ) to the homology of  $\mathcal{K}^d$ . Vassiliev has given a very combinatorial description of the  $E_1$ -term of that spectral sequence. In his thesis, Turchin gave a clever interpretation of these combinatorial data as the Hochschild homology of

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the Poisson operad. We only cite his theorem and refer the reader to [12, Section 5] for details:

**Theorem 1** (Turchin). *The  $E_1$ -term of the Vassiliev spectral sequence computing the homology of  $\mathcal{K}^d$  is isomorphic to the Hochschild homology of the Poisson operad with bracket of degree  $d - 1$ .*

Notice that this theorem has been reproved by Sinha in [11, Corollary 1.3] using the Goodwillie approach that we will describe below and that this alternative approach avoids the combinatorial study of the Vassiliev  $E_1$ -term.

**The Goodwillie approach.**

A different approach to the study of spaces of embeddings has been suggested by Goodwillie and developed with his collaborators Klein and Weiss. A good survey can be found in [4] and with more details in [14] and [5]. As a special case, these authors suggest in [4, Section 5.1] a program to describe spaces of knots as the totalisation of a certain cosimplicial space. This approach has been made completely precise by Sinha in [10] with an elegant formulation in [11]. See also [7] for a very short survey.

The rough idea of this cosimplicial approach is as follows. Consider the configuration space of  $k$  points in  $\mathbf{R}^d$ ,

$$C(k, \mathbf{R}^d) = \{x = (x_1, \dots, x_k) \in (\mathbf{R}^d)^k : x_i \neq x_j \text{ for } i \neq j\}.$$

Sinha has proved that on a suitable compactification  $C\langle k, \mathbf{R}^d \rangle$  of that space we can define cofaces maps,  $\delta_i$ , and codegeneracy maps,  $\sigma_j$ , that gives a cosimplicial space

$$C\langle \bullet, \mathbf{R}^d \rangle.$$

Roughly speaking the cofaces  $\delta_i$  correspond to the doubling of the  $i$ -th point  $x_i$  of the configuration  $x$  in a fixed infinitesimal direction. The fundamental result ([11, Corollary 1.2]) is that, for  $d \geq 4$ , the homotopy totalisation of  $C\langle \bullet, \mathbf{R}^d \rangle$  is homotopically equivalent to  $\mathcal{K}^d$ .

**The Bousfield-Kan spectral sequence computing the homology of knot spaces.**

We now consider the Bousfield-Kan cohomology spectral sequence associated to the cosimplicial space  $C\langle \bullet, \mathbf{R}^d \rangle$  with coefficients in a ring  $\mathbf{F}$  for computing  $H^*(\text{Tot}(C\langle \bullet, \mathbf{R}^d \rangle); \mathbf{F})$ . Notice that the compactification  $C\langle k, \mathbf{R}^d \rangle$  is homotopically equivalent to the genuine configuration space  $C(k, \mathbf{R}^d)$ . Therefore the  $E_2$  terms of that spectral sequence is given by the homology of the configuration spaces mod out by the image of the codegeneracies:

$$E_1^{p,q} = H^q(C(-p, \mathbf{R}^d); \mathbf{F}) / \left( \bigoplus_{j=1}^p \text{im} \sigma_j^* \right)$$

and the differential  $d_1$  is induced by the alternate sum of the cofaces. The cohomology of the configuration spaces as well as the maps induced by the cofaces and codegeneracies are easy to describe, thanks to the work of Fred Cohen.

The above leads to a straightforward description of  $E_2 = H(E_1, d_1)$  as the homology of a certain graph complex defined by chord diagrams on the line with a differential corresponding to contracting adjacent points on that line (see for example [10, Section 7].) This can also be used to prove that, for  $d \geq 4$ , this Bousfield-Kan spectral sequence converges to  $H^*(\text{Tot}(C\langle \bullet, \mathbf{R}^d \rangle); \mathbf{F}) = H^*(\mathcal{K}^d; \mathbf{F})$ . Completely analogous results hold for the *homology* Bousfield-Kan spectral sequence.

### Collapsing of the spectral sequences computing the rational homology of spaces of knots

We arrive to our first result

**Theorem 2** (Lambrechts-Volić, [8]). *For  $d \geq 4$ , the rational homology Bousfield-Kan spectral sequence computing  $H_*(\mathcal{K}^d; \mathbf{Q})$  collapses at  $E_2$ .*

In particular, the rational homology of the space of knots is identified to the homology of an explicit graph complex of chord diagrams. Moreover, using that graph complex, Turchin has noticed that up to a regrading the  $E_1$ -term of the Vassiliev spectral sequence coincide with the  $E_2$ -term of the Bousfield-Kan homology spectral sequence. We deduce the following corollary which was conjectured by Vassiliev

**Corollary 1.** *For  $d \geq 4$ , the whole rational Vassiliev spectral sequence computing  $H^*(\mathcal{K}^d; \mathbf{Q})$  collapses at  $E_1$ .*

When  $d = 3$ , Kontsevich's theorem on the realizability of Vassiliev invariants means that the Vassiliev spectral sequence collapses at  $E_1$  along the main diagonal  $\{E^{p, -p}\}$ .

Cattaneo, Cotta-Ramusino, and Longoni [3] construct also a graph complex mapping to the rational cochain algebra of a space related to  $\mathcal{K}^d$ . Our result should imply that this map is a quasi-isomorphism.

Notice that since the space of long knots is an  $H$ -space (by stacking the knots), it has the rational homotopy type of a product of Eilenberg-MacLane spaces and its rational cohomology is a free graded algebra. Therefore the above completely determines the rational homotopy type of  $\mathcal{K}^d$ . Notice also that by Turchin's result, the rational homology of  $\mathcal{K}^d$  is exactly the Hochschild homology of the Poisson operad.

The main ingredient of the proof of Theorem 2 is the fact that the cosimplicial space  $C\langle \bullet, \mathbf{R}^d \rangle$  is *formal* in the sense of rational homotopy theory, as a diagram. This is a consequence of Kontsevich's [6] formality of the configuration space operad and of the fact that the cofaces and codegeneracies are induced by operadic structure maps. However the detailed proof of Theorem 2 is not as straightforward as we could have hope because Kontsevich formality proof treat a compactification of  $C(k, \mathbf{R}^d)$  that is a manifold with corners, which is not the case for the compactifications  $C\langle k, \mathbf{R}^d \rangle$  used by Sinha in constructing the cosimplicial space. This leads to some technical difficulties and this is where the hypothesis  $d > 3$  is needed. The collapsing of the spectral sequence is then an easy consequence of the formality of the cosimplicial space. It is unclear whether our techniques can be extended to the case  $d = 3$  which would explain the collapsing of the Vassiliev spectral sequence for  $\mathcal{K}^3$ , extending Kontsevich realizability result (but in any case the problem of the convergence of the spectral sequence when  $d = 3$  is very unclear.)

#### Coformality and homotopy Bousfield-Kan spectral sequence.

The formality of the operad means that the singular chains (or some naturally equivalent functor) of the operads are related by a chain of natural quasi-isomorphisms to its homology. In rational homotopy theory there is a dual notion of *coformality*. Roughly speaking it can be interpreted as the existence of natural quasi-isomorphisms between the primitive part of the rational chains over the loop space and their rational homotopy groups. We have the following dual version of Kontsevich's formality result.

**Theorem 3** (Arone-Lambrechts-Turchin-Volić,[2]). *The configuration space operad is coformal.*

It is a little bit less straightforward than in the formal case to deduce that

**Corollary 2.** *The rational homotopy Bousfield-Kan spectral sequence computing  $\pi_*(\mathcal{K}^d) \otimes \mathbf{Q}$  collapses at  $E_2$ .*

Part of this  $E_2$ -term has been computed by Scannell and Sinha [9] with the help of a computer but the complexity is growing exponentially.

As a final comment, note that the combination of the formality of the little balls operad and Goodwillie embedding calculus can be applied to the study of the rational homotopy type of more general embedding spaces  $\text{Emb}(M, \mathbf{R}^d)$  for a general smooth manifold  $M$ . See [1] and the report of Arone in the present proceedings.

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# HOMOLOGICAL STABILITY FOR MAPPING CLASS GROUPS OF NON-ORIENTABLE SURFACES

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The work presented in this note is a summary of a talk given at the Isle of Skye in June 2005. This is work in progress. These results will appear in a forthcoming paper.

## 1. STABILITY THEOREM AND CONSEQUENCES

Let  $S = S_{n,r}$  be a non-orientable surface of genus  $n$  with  $r$  boundary components, i.e.  $S$  is the connected sum of  $n$  copies of  $\mathbb{RP}^2$  with  $r$  discs removed. The mapping class group of  $S$  is

$$\mathcal{M}_{n,r} := \pi_0 \text{Diff}(S_{n,r}; \partial),$$

the group of path components of the space of diffeomorphisms of  $S$  which fix its boundary pointwise.

When  $r \geq 1$ , there are stabilization maps  $\mathcal{M}_{n,r} \rightarrow \mathcal{M}_{n+1,r}$ , obtained by gluing a punctured Moebius band (or a twice punctured  $\mathbb{RP}^2$ ) to the surface and extending the diffeomorphisms by the identity on the added part, and  $\mathcal{M}_{n,r} \rightarrow \mathcal{M}_{n,r+1}$ , obtained similarly by gluing a pair of pants. Gluing a disc on the added pair of pants defines a right inverse to the second map. This means in particular that the map  $H_i(\mathcal{M}_{n,r}; \mathbb{Z}) \rightarrow H_i(\mathcal{M}_{n,r+1}; \mathbb{Z})$  is always injective. Our main theorem is the following.

**Theorem 1.** *The map  $H_i(\mathcal{M}_{n,r}; \mathbb{Z}) \rightarrow H_i(\mathcal{M}_{n+1,r}; \mathbb{Z})$  is surjective when  $n \geq 3i$  and injective when  $n \geq 3i + 3$  for any  $r \geq 1$ .*

*The map  $H_i(\mathcal{M}_{n,r}; \mathbb{Z}[\frac{1}{2}]) \rightarrow H_i(\mathcal{M}_{n,r+1}; \mathbb{Z}[\frac{1}{2}])$  is an isomorphism when  $n \geq 2i - 1$  for any  $r \geq 1$ .*

We conjecture that the same result holds with  $\mathbb{Z}[\frac{1}{2}]$  replaced by  $\mathbb{Z}$  and furthermore that the map  $H_i(\mathcal{M}_{n,r}; \mathbb{Z}) \rightarrow H_i(\mathcal{M}_{n+1,r-1}; \mathbb{Z})$ , induced by gluing a Moebius band, is also an isomorphism stably for any  $r \geq 1$ .

A similar theorem for mapping class groups of *orientable* surfaces, entirely with integral coefficients, was obtained by Harer [4] (improved by Ivanov [6]). Harer moreover shows that, rationally, the best possible homological stability of orientable mapping class groups is of the form  $g \geq \frac{3}{2}i + c$ , where  $g$  is the genus of the surface,  $i$  is the dimension of the homology and  $c$  is a constant [3]. Note that a genus in the orientable case corresponds to

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two genus in the non-orientable case in the sense that, according to the classification of surfaces,

$$(\#_g T^2) \# \mathbb{R}P^2 \cong \#_{2g+1} \mathbb{R}P^2$$

where  $T^2$  denotes the torus. This means that our slope 3 for genus stability in the theorem may be optimal.

Korkmaz calculated  $H_1(M_{n,0})$  [7]. He shows that the group stabilizes when  $n \geq 7$ , which matches our own stability bound for  $r = 1$  according to the conjectural statement including the case  $r = 0$ . Our surjectivity bound also matches Korkmaz’s calculation.

The motivation for the present work comes from a recent result of Madsen and Weiss. In [9], Madsen and Weiss compute the homotopy type of the classifying space of the stable orientable mapping class group  $\Gamma_\infty$  after plus construction, showing that

$$\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega^\infty \mathbb{C}P_{-1}^\infty$$

where  $\Omega^\infty \mathbb{C}P_{-1}^\infty$  is the infinite loop space of a Thom spectrum associated to minus the tautological line bundle over  $\mathbb{C}P^\infty$ .

The work of Madsen-Weiss gives more generally a way to relate the diffeomorphism group of an  $n$ -dimensional manifold, or a subgroup of it preserving a structure on the tangent bundle like an orientation or a spin structure, to an infinite loop space build out of the grassmannian classifying its tangent bundle, provided that there is an appropriate homological stability theorem for the diffeomorphism group. In the case of surfaces, orientable or not, Earle-Eells and Schatz show that the components of the diffeomorphism groups are contractible (except for a few low genus cases) [1, 2], which means that it is enough to have a homological stability theorem for the corresponding mapping class groups. Using the Madsen-Weiss machinery, our Theorem 1 yields the following result.

**Theorem 2.**  $H_i(\mathcal{M}_{n,r}, \mathbb{Z}[\frac{1}{2}]) \cong H_i(\Omega^\infty \Sigma_0^\infty(BO(2)_+); \mathbb{Z}[\frac{1}{2}])$  when  $n \geq 3i + 3$  and  $r \geq 1$ .

Here  $\Omega^\infty \Sigma_0^\infty(BO(2)_+)$  denotes the 0th component of the infinite loop space of the suspension spectrum of  $BO(2)$  with an added basepoint.

Let  $\mathcal{M}_\infty = \text{colim}_{n \rightarrow \infty} (\mathcal{M}_{n,1} \rightarrow \mathcal{M}_{n+1,1} \rightarrow \dots)$  be the stable non-orientable mapping class group. An immediate corollary of Theorem 2 is the non-orientable analogue of the Mumford conjecture.

**Corollary 3.**  $H^*(\mathcal{M}_\infty, \mathbb{Q}) \cong \mathbb{Q}[\lambda_1, \lambda_2, \dots]$  with  $|\lambda_i| = 4i$ .

The classes  $\lambda_i$  are characteristic classes for non-oriented surface bundles, analogues of the Mumford-Morita-Miller classes in the oriented case. Morita’s construction of the MMM-classes [10] (or Madsen-Tillmann’s reinterpretation of it [8]) adapts easily to non-oriented bundles: to a bundle  $E \rightarrow B$  with fiber a —possibly non-orientable— surface, is associate a vertical tangent bundle  $TE \rightarrow E$  whose fiber over a point  $e \in E$  is the tangent

plane to the surface in which  $e$  lies, at the point  $e$ . This 2-plane bundle has a first Pontrjagin class  $p_1 \in H^4(E)$ . Define a class  $\lambda_i \in H^{4i}(B)$  by  $\lambda_i = \tau^*(p_1^i)$ , where  $\tau^* : H^*(E) \rightarrow H^*(B)$  is the Becker-Gottlieb transfer. If the original bundle was orientable, the class  $\lambda_i$  is equal to  $\kappa_{2i}$ , the  $2i$ th MMM-class obtained by choosing an orientation. Corollary 3 says that these are all the stable rational characteristic classes for non-oriented surface bundles.

Let  $G_n$  denote the grassmannian of (non-oriented) 2-planes in  $R^{n+2}$  and let  $U_n^\perp$  denote the  $n$ -plane bundle orthogonal to the tautological bundle over  $G_n$ . Let  $Th(U_n^\perp)$  denote the Thom space of  $U_n^\perp$ . Pulling back  $U_{n+1}^\perp$  along the inclusion  $G_n \hookrightarrow G_{n+1}$  gives maps

$$S^1 \wedge Th(U_n^\perp) \rightarrow Th(U_{n+1}^\perp)$$

defining a Thom spectrum which we denote  $Th(U^\perp)$ . A version of Theorem 1 completely with integer coefficients combined with the Madsen-Weiss machinery would yield the following

**Conjecture.**  $H_*(\mathcal{M}_\infty; \mathbb{Z}) \cong H_*(\Omega_0^\infty Th(U^\perp); \mathbb{Z})$

## 2. COMPLEXES OF ARCS

To prove Theorem 1, we use complexes of arcs in surfaces. Let  $S$  be a surface, orientable or not, and let  $\vec{\Delta}$  be a set of *oriented points* in  $\partial S$ , that is each point comes with the choice of an orientation of the component of  $\partial S$  it lies in. If we choose an orientation for each boundary component of  $S$ , there are two types of points in each boundary: those with positive orientation, i.e. the same as the chosen one, and those with negative orientation. If  $S$  is orientable, a choice of orientation for  $S$  gives a choice of orientation of its boundary components and thus a decomposition  $\vec{\Delta} = \Delta^+ \cup \Delta^-$ , where each subset represents an orientation class of points.

We assume that  $(S, \vec{\Delta})$  is *non-orientable* in the sense that either  $S$  is non-orientable, or it has no orientation compatible with the orientations of the points of  $\vec{\Delta}$  on its boundary, i.e. if  $S$  is orientable, then  $\Delta^+$  and  $\Delta^-$  are non-empty.

We consider arcs embedded in  $S$  and intersecting  $\partial S$  transversally at its endpoint (and not otherwise).

**Definition 1.** A 1-sided arc in  $(S, \vec{\Delta})$  is an arc embedded in  $S$  with boundary in  $\vec{\Delta}$  whose normal bundle identifies the orientations of its end points. An arc is non-trivial if it is not isotopic, fixing its endpoints, to an arc of  $\partial S$  disjoint from  $\vec{\Delta}$ .

Figure 1 gives a few examples of 1-sided arcs. Note that an arc with boundary on a single point of  $\vec{\Delta}$  is 1-sided if and only if its normal bundle is a Moebius band, so that it actually defines a 1-sided curve in  $S$ . On

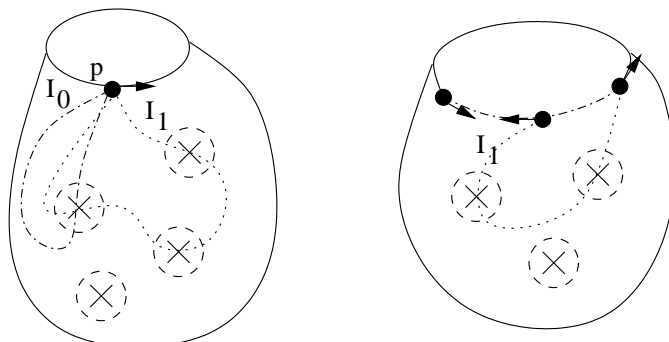


FIGURE 1. 1-sided arcs. (The crosses represent Moebius bands.)

the other hand, if  $S$  is orientable, the 1-sided arcs are the arcs with one boundary point in  $\Delta^+$  and the other one in  $\Delta^-$ .

We want to show high connectivity of a complex of 1-sided arcs in the usual meaning of the word. To do this, we need the more general definition given above because of the following phenomenon: one can have two 1-sided arcs  $\langle I_0, I_1 \rangle$  based at a point  $p$  of a non-orientable surface  $S$  so that  $I_1$  is 2-sided as an arc of  $S \setminus I_0$ . However  $I_1$  is still 1-sided in the sense of Definition 1 if one keeps track of the orientation of the point  $p$  when cutting along  $I_0$ . (See Fig. 1). Note that a surface can actually become orientable after cutting along a single 1-sided curve.

**Definition 2.** Let  $\mathcal{F}(S, \vec{\Delta})$  be the simplicial complex whose vertices are isotopy classes of non-trivial 1-sided arcs in  $(S, \vec{\Delta})$  and where a collection of arcs  $\langle I_0, \dots, I_p \rangle$  forms a  $p$ -simplex if the arcs are embeddable disjointly —except possibly at the endpoints— and are not pairwise isotopic.

By the above remark, when  $S$  is orientable, the simplicial complex  $\mathcal{F}(S, \vec{\Delta})$  is isomorphic to Harer’s complex  $BZ(S; \Delta^+, \Delta^-)$  of arcs joining points of  $\Delta^+$  to points of  $\Delta^-$  [4]. Harer shows that this complex is highly connected. We extend his result to  $\mathcal{F}(S, \vec{\Delta})$  for any surface  $S$ .

To state the theorem, we need to define a few parameters. Let  $h$  denote the genus of  $S$  if  $S$  is non-orientable and twice its genus if  $S$  is orientable. Let  $r$  be the number of boundary components of  $S$  and  $r'$  the number of boundaries with points of  $\vec{\Delta}$ . The points of  $\vec{\Delta}$  define edges in  $\partial S$ . Define  $m$  to be the number of orientable edges, i.e. edges whose endpoints have the same orientation, and define  $l$  to be half the number of non-orientable edges. (As the components of  $\partial S$  are circles, there is always an even number of non-orientable edges.)

**Theorem 4.**  $\mathcal{F}(S, \vec{\Delta})$  is  $(2h + r + r' + l + m - 6)$ -connected.

We also need the analogous theorem for the other, more obvious, generalization of Harer’s complex  $BZ$  to non-orientable surfaces. Namely, let

$\Delta_0$  and  $\Delta_1$  be two non-empty, disjoint sets of points in  $\partial S$  and define  $BZ(S; \Delta_0, \Delta_1)$  to be the simplicial complex whose vertices are isotopy classes of non-trivial arcs  $I$  in  $S$  with  $\partial_0 I \in \Delta_0$  and  $\partial_1 I \in \Delta_1$ . A simplex is a collection of such arcs embeddable disjointly and not pairwise isotopic.

**Theorem 5.**  $BZ(S; \Delta_0, \Delta_1)$  is  $(2h + r + r' + l + m - 6)$ -connected, where  $m$  is now the number of pure edges, i.e. those with both boundary point in the same set, and  $l$  is half the number of impure edges, i.e. those connecting points of  $\Delta_0$  to points of  $\Delta_1$ .

Theorems 4 and 5 are two generalizations of [4, Thm.1.6]. The proofs of Theorems 4 and 5 are very similar, but do not depend on [4]. They both rely on the high connectivity (actually contractibility in most cases) of the complex of all arcs with boundary points in a set of points  $\Delta$  in  $\partial S$ , where  $\Delta$  in the first case is obtained from  $\vec{\Delta}$  by forgetting the orientations, and in the second case  $\Delta$  is the union of  $\Delta_0$  and  $\Delta_1$ . A very simple proof of this last result is given in [5] and we exploit further the same techniques. (Harer proves the same result in the orientable case using Thurston's theory of train tracks [4, Thm.1.5].)

From Theorems 4 and 5, it is easy to deduce high connectivity of the corresponding subcomplexes of arc systems with connected complements. In the case of oriented points, we actually need a slightly better complex.

**Definition 3.** Let  $\mathcal{G}(S, \vec{\Delta})$  be the subcomplex of  $\mathcal{F}(S, \vec{\Delta})$  whose  $p$ -simplices are the simplices  $\langle I_0, \dots, I_p \rangle$  of  $\mathcal{F}(S, \vec{\Delta})$  such that  $S \setminus \{I_0, \dots, I_p\}$  is connected and non-orientable (unless it is of genus 0).

**Theorem 6.**  $\mathcal{G}(S, \vec{\Delta})$  is  $(\lceil \frac{n}{2} \rceil - 2)$ -connected.

Theorem 1 is then proved via a spectral sequence argument. For the first map, we use the action of  $\mathcal{M}_{n,r}$  on  $\mathcal{G}(S_{n,r}, \vec{\Delta})$  with  $\vec{\Delta}$  a single point, and for the second map we use the action of  $\mathcal{M}_{n,r}$  on the subcomplex of  $BZ(S_{n,r}; \Delta_0, \Delta_1)$  of collections of arcs with connected complement, where  $\Delta_0$  and  $\Delta_1$  are two points in different boundary components of  $S$ . The coefficient  $\mathbb{Z}[\frac{1}{2}]$  in the second case comes from the fact that the action is not transitive on the vertices of the second complex.

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