Non-renewable energy resources as input for physical capital accumulation: A new approach.*

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Abstract

In contrast to the standard approach of energy economics, this paper assumes that physical capital accumulation is relatively more energy-intensive than consumption. By means of Gaussian hypergeometric functions we provide a closed-form representation of the optimal solution paths of our variables in levels, whatever the initial conditions (*i.e.*, global dynamics). We find that, in general, the optimal trajectories are non-monotonic. Dasgupta and Heal (1974) pointed out this result (local dynamics) for the optimal consumption in a model with identical technology for both physical capital accumulation and consumption. However, our paper introduces five novelties with respect to their study, namely, global dynamics, the importance of the proportion of non-renewable energy resources to endowment of physical capital ($S(0)/K(0)$), the role of the technical progress, U-shape behaviour of consumption, and non-monotonicity of resource extraction.

Keywords: Non-renewable resources, Energy-saving technical progress, Special functions, Global dynamics.

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1 Introduction

Energy economics widely admits that physical capital accumulation is a key component in offsetting the constraint on production possibilities due to the usage of non-renewable energy resources. The standard approach (see for instance Dasgupta and Heal (1974, 1979), Solow (1974a,b), Stiglitz (1974), Hartwick (1989) and Pezzey and Withagen (1998)) assumes the same technology for both physical capital accumulation and consumption, which implies (among other things) that the energy intensity of both sectors is the same. However, data does not support this simplification. In fact, taking the Structural Analysis Database of OECD and the Energy Balances and Energy Prices and Taxes of IEA, Azomahou *et al.* (2006) build an energy intensity measurement (ratio between energy consumption and value added) of 14 sectors of the economy.[1] One observes that this ratio is particularly high for activities closely related to physical capital accumulation (ex: iron and steel sector (0.809), transport and storage (0.85), non-ferrous metals (0.599), and non-metallic minerals (0.507). However, for consumption goods energy intensity is clearly lower (ex: food and tobacco (0.134), textile and leather (0.082), and construction (0.018)). Therefore, data, contrary to the standard assumption, seems to support that physical capital accumulation is more energy-intensive than consumption.

In this paper, we study the implications for economic growth in assuming a different technology for each sector, physical capital accumulation and consumption. More precisely, in an economy where energy is directly produced by extracting non-renewable resources, we assume that physical capital accumulation is relatively more energy-intensive than consumption. We use a general equilibrium model with three sectors: consumption, physical capital and extraction. In addition, this paper pays special attention to energy-saving technologies defined as technologies that reduces energy intensity. Indeed, the importance of these kinds of technologies were pointed out in numerous studies (see for instance Boucekkine and Pommeret (2004), COM (2005) and Pérez-Barahona and Zou (2006a,b)).

Our primary focus is on the analysis of the balanced growth path equilibrium (BGP). By means of Gaussian hypergeometric functions we provide a closed-form representation of the optimal solution paths of our variables in levels. In contrast to the standard approach, this method of Special functions allows for a global description of the dynamics of the economy[2] and does not rely on dimension reduction (see Abramowitz and Stegun (1970) and Boucekkine and Ruiz-Tamarit (2008) for further details). Therefore, we present a closed-form representation of the optimal trajectories whatever the initial conditions, *i.e.*, we do not limit our study to the values of initial conditions in the neighbourhood of an equilibrium point. We also prove that there is asymptotic convergence to the BGP. Moreover, we explicitly study the monotonicity of the optimal trajectories of the variables in levels, which is typically unfeasible in the standard approach. We get that, in general, the optimal trajectories are non-monotonic. Dasgupta and Heal (1974) previously observed this result (local dynamics) for the optimal consumption in a model with the same technology for both physical capital accumulation and consumption. However, our paper raises several novelties with respect their study, where these new results are a direct consequence of assuming both that physical capital accumulation sector is relatively more energy intensive than consumption, and growing technical progress (in particular, energy-saving). Actually, Dasgupta and Heal (1974) showed that an economy with large endowment of both non-renewable energy resources and physical capital initially increases consumption but, after some time, it will decrease. However, our model points out the possibility of an opposite behaviour of consumption. Indeed, under both a relative scarcity of non-renewable energy resources with respect to the endowment of physical capital and low growth rates of technical progress, consumption depicts a U-shape. Moreover, we prove that the more impatient the household, the greater is the number of periods where consumption decreases. Finally, in contrast with the same authors, we show that resource extraction is not necessarily monotonic.

The remainder of the paper is organized as follows. Section 2 describes our economy, providing the dynamical system corresponding to the optimal solution. In Section 3 we study the BGP equilibrium. Section 4 solves the dynamical system presented in Section 2 by using Gaussian hypergeometric functions. In Section 5 we provide the analytical expressions of the optimal solution paths. Section 6 studies the monotonicity properties of the optimal trajectories in levels. Finally, we consider some concluding remarks in Section 7.

2 The model

Based on Dasgupta and Heal (1974) and Hartwick (1989), let us consider a three sector economy with non-renewable energy resources, where the population is constant.[3] The final good sector produces a non-durable good, which can be assigned to consumption or investment. This final good is produced by means of a linear technology, where the physical capital (durable good) is the sole input. The durable good sector accumulates physical capital by means of a technology defined over two inputs: investment and energy. Finally, the extraction sector directly produces energy by extracting non-renewable energy resources from a given stock.

2.1 Final good sector

The final good sector produces a non-durable good $Y(t)$ by means of the following linear technology:

$$Y(t) = A(t)K(t), \tag{1}$$

where $A(t)$ is the disembodied technical progress[4], and $K(t)$ represents the physical capital that is the sole input to produce the final good. Notice that our model assumes an AK function for final good production. We consider this linear technology for the sake of simplicity. However, following Barro and Sala-i-Martin (2003) and Groth and Schou (2007), this assumption is reasonable if we consider a broad concept of physical capital, which includes human capital, knowledge, public infrastructure, etc.[5] Final good is used either to consume, $C(t)$, or to invest in physical capital, $I(t)$, verifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t). \tag{2}$$

2.2 Durable good sector

The durable good sector accumulates physical capital, $\dot{K}(t)$, by using two inputs. On the one hand, the durable good sector takes the fraction of final good devoted to investment, while on the other hand this sector uses the energy, $R(t)$, produced by the extraction sector. The technology for physical capital accumulation is represented by the following Cobb-Douglas function:

$$\dot{K}(t) = [\theta(t)R(t)]^{\alpha} I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0, \tag{3}$$

where $\theta(t)$ denotes the embodied energy-saving technical progress, and $K(0)$ is the initial stock of physical capital, which is considered as given.[6]

At this moment it is worthwhile to consider two observations. First, following the neoclassical growth framework introduced by Ramsey (1928), Cass (1965) and Koopmans (1965), the standard literature on non-renewable energy resources (see for instance Dasgupta and Heal (1974, 1979), Stiglitz (1974), Solow (1974a,b), Hartwick (1989), and Smulders and Nooij (2003)) considers energy resources as an input for the production of final good. Moreover, since they also assume a single production process (final good), the technology for physical capital accumulation, $\dot{K}(t)$, and consumption is the same.[7] As we pointed out in the introduction, an important implication of this assumption is that the energy intensity of both sectors is the same too. However, there is neither conceptual nor empiric support for this hypothesis. Indeed, data goes in the direction of considering that physical capital accumulation is more energy-intensive than consumption (Azomahou *et al.* (2006)). Therefore, in contrast to the standard approach, our paper proposes a set-up with two separate production process, namely, production of final good and production of equipment, where physical capital accumulation is an activity relatively more energy-intensive than consumption. Indeed, we consider that the non-durable good is produced by means of an AK production function (equation (1)), while the technology for physical capital accumulation is a function of two inputs, namely, investment and non-renewable energy resources (equation (3)). Therefore, our model assumes that $\dot{K}(t) = g[I(t), R(t)]$, which contrasts with the standard law of motion of physical capital, *i.e.*, $\dot{K}(t) = I(t)$. Finally, as a second observation, we should notice that, in order to get analytical expressions, our model considers a Cobb-Douglas functional form for the production of equipment (equation (3)). As it is clear, this functional form assumes substitutability between energy and investment. However, this hypothesis can be justified by following a similar argument as in Dasgupta and Heal (1979), where one can assume substitutability if $I(t)$ is interpreted as final good service. Therefore, we consider that the provision of a flow of final good $I(t)$ implies the provision of a certain energy flow.[8]

2.3 Extraction sector

Energy is directly produced by extracting non-renewable energy resources, $R(t)$, from a homogeneous stock $S(t)$. As Dasgupta and Heal (1974) and Hartwick (1989), we assume costless extraction.[9] The law of motion of the stock of non-renewable energy resources is described by the expression

$$R(t) = -\dot{S}(t), \tag{4}$$

where $S(0)$ is the endowment of non-renewable energy resources of the economy and it is considered as given. Since we can not extract more than the available stock, the following restriction should be included:

$$S(t) \geq 0. \tag{5}$$

2.4 Optimal solution

The central planner (optimal solution) maximizes the instantaneous utility function of the representative household:

$$\max \int_0^\infty \ln[C(t)] \exp(-\rho t) dt, \text{ with } \rho > 0,$$

subject to the equations (1)-(5), where ρ is the time preference parameter (it is assumed to be a positive discount factor).[10]

We can rewrite the previous problem as the following optimal control problem with mixed constraints:

$$\max \int_0^\infty \ln[A(t)K(t) - I(t)] \exp(-\rho t) dt, \text{ with } \rho > 0$$

subject to:

$$\dot{K}(t) = [\theta(t)R(t)]^\alpha I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0,$$

$$R(t) = -\dot{S}(t),$$

$$S(t) \geq 0,$$

$$\text{with } K(0) \text{ and } S(0) \text{ given,}$$

where $K(t)$ and $S(t)$ are the state variables, and $I(t)$ and $R(t)$ are the control variables. Following Sydsæter *et al.* (1999):

Proposition 1. *Given the initial conditions $K(0)$ and $S(0)$, the solution of our optimal control problem is a path $\{C(t), R(t), I(t), S(t), K(t)\}$ that satisfies the following conditions:*

$$\dot{K}(t) = \theta(t)^\alpha R(t)^\alpha I(t)^{1-\alpha}, \tag{6}$$

$$\frac{\dot{C}(t)}{C(t)} = (1-\alpha)A(t)\theta(t)^\alpha \left(\frac{R(t)}{I(t)}\right)^\alpha - \alpha \left(\frac{\dot{\theta}(t)}{\theta(t)} - \frac{\dot{I}(t)}{I(t)} + \frac{\dot{R}(t)}{R(t)}\right) - \rho, \tag{7}$$

$$(1-\alpha)A(t)\theta(t)^\alpha \left(\frac{R(t)}{I(t)}\right)^\alpha = \alpha\frac{\dot{\theta}(t)}{\theta(t)} + (1-\alpha)\left(\frac{\dot{I}(t)}{I(t)} - \frac{\dot{R}(t)}{R(t)}\right), \tag{8}$$

$$A(t)K(t) = C(t) + I(t), \tag{9}$$

$$\dot{S}(t) = -R(t). \tag{10}$$

This system has five equations and five unknowns.[11] Equation (6) is the accumulation law of physical capital. Notice that, in this model, energy is an input to accumulate physical capital. If we denote $\dot{K}(t) = g[\theta(t), I(t), R(t)]$, for a general utility function $U[C(t)]$, equation (7) can be rewritten as

$$\frac{U_{cc}\dot{C}}{U_c} - \rho = \frac{\dot{g}_I}{g_I} - Ag_I.$$

This is the Ramsey optimal savings relation. Similarly, equation (8) is equivalent to the following expression:

$$A(t)g_I = \frac{\dot{g}_R}{g_R},$$

which is the efficient condition for using up the non-renewable resource, *i.e.*, the Hotelling rule. Finally, equation (9) is the budget constraint of the economy, while equation (10) is the law of motion of the stock of non-renewable energy resources.

3 Balanced growth path equilibrium

Let us define BGP as the situation where all the endogenous variables grow at a constant rate, *i.e.*, $x(t) = \overline{x}\exp(\gamma_x t)$, where γ_x is the growth rate of $x(t)$ and $\overline{x}$ the corresponding level. Following Solow (1974a), we assume that $A(t) = \overline{A}\exp(\gamma_A t)$ and $\theta(t) = \overline{\theta}\exp(\gamma_\theta t)$ for all t, where $\overline{A}, \overline{\theta}, \gamma_A, \gamma_\theta$ are considered as strictly positive and exogenous parameters. Then, from the dynamical system (6)-(10), it is easy to get the growth rates of the endogenous variables of our model along the BGP (necessary condition):[12]

Proposition 2. *Along the BGP, $Y(t)$, $C(t)$, and $I(t)$ grow at rate*

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho,$$

the growth rate of the stock of physical capital is

$$\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho,$$

and extraction, $R(t)$, and stock of non-renewable energy resources, $S(t)$, grow at rate

$$\gamma_R = \gamma_S = -\rho.$$

As we observed in Section 2.4, the parameter ρ is the (positive) discount rate that represents the degree of impatience of the household. Indeed, the greater ρ the more impatient the household is. Therefore, households prefer a higher level of present consumption than postponing it. Indeed, from Proposition 2, we can observe that the greater ρ the greater is the decreasing rate of the stock of non-renewable energy resources. The reason is the following. Since the household does not want to postpone consumption, it prefers to extract a greater amount of energy resource in order to get higher present consumption.

Furthermore, since a greater ρ induces a greater extraction, this reduces the growth rates of the economy along the BGP for a given growth rate of technical progress. However, this negative effect of the degree of impatience can be compensated for greater growth rates of both disembodied (γ_A) and energy-saving technical progress (γ_θ). More precisely, Proposition 2 establishes that if the growth rate of technical progress is high enough ($\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A > \rho$), then $\gamma_Y(= \gamma_C = \gamma_I) > 0$ and $\gamma_K > 0$ (if BGP exists). The reason is that technical progress (in particular, energy-saving) reduces the energy intensity of the economy (*i.e.,* increases energy efficiency). Indeed, if we define energy intensity as the ratio $R(t)/Y(t)$, it is easy to prove that, in our model, the growth rate of energy intensity along the BGP is

$$\gamma_{R/Y} = -\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A\right),$$

which is negative for $\gamma_A > 0$ and $\gamma_\theta > 0$. Moreover, this is consistent with empirical works such as COM (2005) and Azomahou *et al.* (2006).

4 Analytical solution: Gaussian hypergeometric representation

The aim of this section is to provide an analytical characterization of the dynamics of the model. As we explained in Section 1, the equilibrium of our economy is described by the dynamical system (6)-(10). By means of a family of Special functions called Gaussian hypergeometric, we obtain closed-form solutions of the optimal paths. Special functions are frequently used in mathematical physics and computational mathematics (see Luke (1969), Abramowitz and Stegun (1972), Temme (1996) and Andrews *et al.* (1999) for a revision of the properties and applications of these functions). In this paper, we have to deal with a similar calculus problem as in Boucekkine and Ruiz-Tamarit (2008) (page 40, Proposition 1), where Gaussian hypergeometric functions are applied to solve the problem. Therefore, in this paper we follow their resolution method.[13]

Let us start with a brief description of how to solve recursively the dynamical system (6)-(10).[14] By defining the ratio $X(t) = R(t)/I(t)$, it is easy to verify that the equation (8) is a Bernoulli's equation (see Appendix A). Therefore, following Sydsæter *et al.* (1999), we get the analytical solution for the ratio $X(t)$. Taking the solution of $X(t)$ into equation (7), we get $C(t)$ as solution of the corresponding linear first-order differential equation. Similarly, by rewriting equation (6) as $\dot{K}(t) = \theta(t)^\alpha X(t)^\alpha [A(t)K(t) - C(t)]$, and taking the previously obtained formulas for $X(t)$ and $C(t)$, $K(t)$ is calculated as the solution of a linear first-order differential equation. Knowing $C(t)$ and $K(t)$, the analytical expression for $I(t)$ is straightforward from equation (9). Moreover, by retrieving the ratio $X(t)$, we get the formula for $R(t)$. Finally, from equation (10), S(t) is obtained by solving another linear first-order differential equation.

4.1 Optimal solution paths

By solving recursively the dynamical system, together with the parameter constraints involved in both the transversality conditions and the Gaussian hypergeometric representation (see Appendix A for details), we establish the optimal solution paths of our economy.

As we will see in this section, in general, the optimal solution paths ($x(t) = C(t), K(t), I(t), Y(t), R(t)$ or $S(t)$) are functions in the form of

$$x(t) = f(\text{parameters}, K(0), S(0), \varphi, t),$$

where φ is the solution of a non-linear equation (see Appendix A, equation (A.25)):[15]

$$h(\text{parameters}, K(0), S(0), \varphi) = 0.$$

4.1.1 Optimal solution paths for consumption $C(t)$

From equation (A.10) (see Appendix A), we get $C(t)$. Then, by rearranging terms we establish the following proposition:

Proposition 3. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $C(t)$, starting from*

$$C(0) = \frac{\alpha \overline{A}}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A} \theta^\alpha}\right)} \frac{Z + \rho}{Z} K(0),$$

such that

$$C(t) = C(0)\left[\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)\exp(-Zt) + \frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right]^{\frac{1}{\alpha}}$$

$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\}; \tag{11}$$

(ii) *this equilibrium path shows transitional dynamics, approaching the unique BGP asymptotically*

$$C_{BGP}(t) = C(0)\left(\frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

where $K(0)$ is given.

Regarding the sign of the optimal solution path, for $C(0) > 0$, condition (A.26) implies that both the transitional dynamics and BGP are positive. From Proposition 3, the condition for $C(0) > 0$ is that ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 0$. Then, we establish the following corollary:

Corollary 1. *Under conditions (A.22), (A.26), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 0$, the optimal solution path for $C(t)$ is strictly positive for all $t \geq 0$.*

4.1.2 Optimal solution paths for physical capital $K(t)$

Plugging equation (A.19) into equation (A.12) and rearranging terms (see Appendix A), we obtain the optimal solution path for physical capital:

Proposition 4. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $K(t)$, starting from $K(0)$, such that*

$$K(t) = K(0)(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)} \left(\frac{\varphi Z}{\exp(Zt)} + \alpha\overline{A\theta}^{\alpha}\right)^{\frac{1}{\alpha}}$$

$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right)t\right\}; \tag{12}$$

(ii) *this equilibrium path shows transitional dynamics, approaching the unique BGP* *asymptotically*

$$K_{BGP}(t) =$$

$$K(0)\left(\frac{\alpha\overline{A}\overline{\theta}^{\alpha}}{\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\exp\left\{\left(\gamma_\theta+\frac{1-\alpha}{\alpha}\gamma_A-\rho\right)t\right\};$$

where $K(0)$ is given.

Furthermore, as in the previous case, we establish the following corollary to ensure a positive optimal solution path:

Corollary 2. *For a given $K(0) > 0$, under conditions (A.22), (A.26), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right) > 0$, the optimal solution path for $K(t)$ is strictly positive for all $t \geq 0$.*

The proof is provided upon request (see Appendix E, Proof of Corollary 2).

4.1.3 Optimal solution paths for investment $I(t)$

Substituting equation (A.19) into equation (A.13) (see Appendix A) and rearranging terms, we find the following proposition:

Proposition 5. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $I(t)$, starting from*

$$I(0) = \overline{A}K(0)\left(1-\alpha\frac{Z+\rho}{Z}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\right),$$

such that

$$I(t) = \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\left(\frac{\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}{(\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha})\exp(Zt)}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right)-\alpha\frac{Z+\rho}{Z}\right]\exp\left\{\left(\gamma_\theta+\frac{1}{\alpha}\gamma_A-\rho\right)t\right\}; \quad (13)$$

(ii) *this equilibrium path shows transitional dynamics, approaching the unique BGP asymptotically*

$$I_{BGP}(t) = \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\left(\frac{\alpha\overline{A}\overline{\theta}^{\alpha}}{\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\left(1-\frac{Z+\rho}{Z}\alpha\right)$$

$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

where $K(0)$ is given.

It is easy to prove that condition (A.22) implies $\alpha\frac{Z+\rho}{Z} < 1$. If ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\right) > 1$, then $I(0) > 0$. Moreover, as in Corollary 2, we prove that ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha \exp(Zt)}\right) > {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\right)$. Therefore, equation (13) will also be positive. Hence, we establish the following corollary:

Corollary 3. *For a given $K(0) > 0$, under conditions (A.22), (A.26), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\right) > 1$, the optimal solution path for $I(t)$ is strictly positive for all $t \geq 0$.*

4.1.4 Optimal solution paths for output $Y(t)$

Since $Y(t) = C(t) + I(t)$, Propositions 3 and 5 provide the optimal solution path for output:

Proposition 6. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $Y(t)$, starting from $Y(0) = \overline{A}K(0)$, such that*

$$Y(t) = \overline{A}K(0)\left(\frac{\varphi Z + \alpha\overline{A\theta}^\alpha \exp(Zt)}{(\varphi Z + \alpha\overline{A\theta}^\alpha)\exp(Zt)}\right)^{\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha \exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\right)}$$

$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\}; \tag{14}$$

(ii) *this equilibrium path shows transitional dynamics, approaching the unique BGP asymptotically*

$$Y_{BGP}(t) = Y(t) = \overline{A}K(0)\left(\frac{\alpha\overline{A\theta}^\alpha}{\varphi Z + \alpha\overline{A\theta}^\alpha}\right)^{\frac{1}{\alpha}} \frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\right)}$$

$$\cdot \exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

where $K(0)$ is given.

Concerning the sign of the optimal solution path, we proceed as in Corollary 2:

Corollary 4. *For a given $K(0) > 0$, under conditions (A.22), (A.26), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right) > 0$, the optimal solution path for $Y(t)$ is strictly positive for all $t \geq 0$.*

4.1.5 Optimal solution paths for extraction of non-renewable resources $R(t)$

Since $R(t) = X(t)I(t)$, from equations (A.13), (A.19) (see Appendix A) and (B.1) (see Appendix B) we get the optimal solution path for the extraction of non-renewable resources:

Proposition 7. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $R(t)$, starting from*

$$R(0) = \overline{A}K(0)\left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}\left(1 - \alpha\frac{Z+\rho}{Z}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}\right),$$

such that

$$R(t) = \left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}\frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right) - \alpha\frac{Z+\rho}{Z}\right] \cdot \exp(-\rho t); \tag{15}$$

(ii) *this equilibrium path shows transitional dynamics, approaching the unique BGP asymptotically*

$$R_{BGP}(t) = \left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}\frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}\left[1 - \alpha\frac{Z+\rho}{Z}\right] \cdot \exp(-\rho t);$$

where $K(0)$ is given.

Following the same reasoning as in Corollary 3, we can establish the following condition for a positive optimal solution path for $R(t)$:

Corollary 5. *For a given $K(0) > 0$, under conditions (A.22), (A.26), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 1$, the optimal solution path for $R(t)$ is strictly positive for all $t \geq 0$.*

4.1.6 Optimal solution paths for the stock of non-renewable resources $S(t)$

From Appendix A, substituting equation (A.21) into equation (A.15) and rearranging terms, we get the optimal solution path for the stock of non-renewable energy resources:

Proposition 8. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $S(t)$, starting from $S(0)$, such that*

$$S(t) = C(0)\left(\frac{Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$
$$\cdot \left\{ \frac{\left(\frac{Z}{Z+\rho}\right)^2 \frac{1}{\alpha}\frac{1}{Z+\Omega}}{\exp\left((Z+\Omega)\frac{Z+\rho}{Z}t\right)} {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}\exp(Zt)}\right) - \frac{1}{\rho}\frac{1}{\exp(\rho t)} \right\}$$
$$\cdot \exp(-\rho t); \tag{16}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to zero;*

where $K(0)$ and $S(0)$ are given.

Regarding the sign of the optimal solution path for the stock of non-renewable resources, for $C(0) > 0$, it is enough to require that $\exp(\rho t)$ increases faster than $\exp\left((Z+\Omega)\frac{Z+\rho}{Z}t\right)$:

Corollary 6. *For $K(0) > 0$ and $S(0) > 0$ given, under conditions (A.22), (A.26), ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$, and $\rho > (Z+\Omega)\frac{Z+\rho}{Z}$, the optimal solution path for $S(t)$ is strictly positive for all $t \geq 0$.*[16]

Finally, we conclude this section by summarizing all the conditions to guarantee strictly positive optimal solution paths:

Corollary 7. *For $K(0) > 0$ and $S(0) > 0$ given, under conditions (A.22), (A.26), ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 1$, $\rho > (Z+\Omega)\frac{Z+\rho}{Z}$ and , the optimal solution paths for*

$X(t)$, $C(t)$, $K(t)$, $I(t)$, $Y(t)$, $R(t)$, and $S(t)$, are strictly positive for all $t \geq 0$.

5 On the monotonicity of the optimal solution paths

An important outcome of having closed-form solutions is that we can explicitly study the monotonicity of the optimal trajectories of the variables in levels, whatever the initial conditions. Indeed, this study is typically unfeasible in the standard approach. This section will show that, in general, the optimal trajectories are non-monotonic. Dasgupta and Heal (1974) already pointed out this result (local dynamics) for the optimal consumption in a model with the same technology for both physical capital accumulation and consumption. However, our paper raises several novelties with respect their study. Also note that, since Dasgupta and Heal (1974)'s result focused on $C(t)$ and $R(t)$, this section will similarly concentrate on the monotonicity properties of these two variables.[17]

As we will see in Sections 5.2 and 5.3, the sign of φ is a key element in determining the monotonicity properties of our optimal solutions paths. Therefore, we have to study this sign before dealing with the monotonicity of the optimal trajectories.

5.1 The relationship between φ and $S(0)/K(0)$

From condition (A.25), taking all parameters as given, we observe that the ratio $S(0)/K(0)$ determines φ (and, in particular, the sign of φ).[18] Let us define $(S(0)/K(0))^*$ as the ratio corresponding to $\varphi = 0$. Then,

$$\left(\frac{S(0)}{K(0)}\right)^* = \alpha\overline{A}\left(\frac{Z}{\alpha\overline{A}\theta^\alpha}\right)^{\frac{1}{\alpha}}\left(\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+\Omega} - \frac{Z+\rho}{Z}\frac{1}{\rho}\right). \tag{17}$$

From (A.25), by applying the implicit function theorem we get that

$$\frac{\partial\varphi}{\partial(S(0)/K(0))} < 0.$$

The proof is provided in Appendix C.

Hence, for $S(0)/K(0) > (<)(S(0)/K(0))^*$, φ is negative (positive).

5.2 Monotonicity of consumption

Equation (11) provides the optimal solution path for consumption. First, let us study detrended consumption ($\tilde{C}(t)$), *i.e.*,

$$\tilde{C}(t) = C(0)\left[\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)\exp(-Zt) + \frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right]^{\frac{1}{\alpha}}. \tag{18}$$

From equation (18), we get

$$\frac{d\tilde{C}(t)}{dt} =$$

$$-\varphi\left\{\tilde{C}(t)\frac{Z\exp(-Zt)}{\alpha}\left(\frac{\varphi Z\exp(-Zt) + \alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{-1}\frac{1}{\varphi + \frac{\alpha\overline{A\theta}^{\alpha}}{Z}}\right\} \tag{19}$$

From Corollary 7, the expression between braces is positive. Thus, the sign of φ determines the sign of the derivative. Hence, we can establish the following proposition:

Proposition 9.

(1) *If* $S(0)/K(0) > (<)(S(0)/K(0))^*$*, then* $\tilde{C}(t)$ *increases (decreases) monotonically for all* t*.*

(2) *If* $S(0)/K(0) = (S(0)/K(0))^*$*, then*

$$\tilde{C}(t) = \alpha\overline{A}\frac{Z+\rho}{Z}K(0)$$

for all t*.*

Knowing the monotonicity properties of $\tilde{C}(t)$ allows us to study $C(t)$. Since $C(t) = \tilde{C}(t)\exp(\gamma_C t)$, where $\gamma_C = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$, it is easy to prove that

$$\frac{dC(t)}{dt} = \exp(\gamma_C t)\left(\frac{d\tilde{C}(t)}{dt} + \tilde{C}(t)\gamma_C\right). \tag{20}$$

From Corollary 7, we observe that all terms in equation (20) are positive except $d\tilde{C}(t)/dt$. Therefore, we can study this derivative from the monotonicity properties of $\tilde{C}(t)$. If $S(0)/K(0) > (S(0)/K(0))^*$, then $d\tilde{C}(t)/dt > 0$ for all t. Consequently, from equation (20), $dC(t)/dt > 0$ for all t. If $S(0)/K(0) = (S(0)/K(0))^*$, then $d\tilde{C}(t)/dt = 0$ for all t. So, from equation (20), we conclude that $dC(t)/dt > 0$ for all t. However, the difficult case emerges for $S(0)/K(0) < (S(0)/K(0))^*$ because $d\tilde{C}(t)/dt < 0$ for all t. Substituting equation (19) into equation (20), and rearranging terms, it yields

$$\frac{dC(t)}{dt} = \exp(\gamma_C t)\tilde{C}(t)\left[\gamma_C - \left(\frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)\right]. \tag{21}$$

As before, all terms of equation (21) are positive. Since $\exp(Zt)$ increases with time, the term between brackets achieves its maximum value when $t = 0$. Thus, if

$$\gamma_C \geq \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}(\equiv \gamma^*),$$

then, $dC(t)/dt > 0$ for all $t > 0$. However, if $\gamma_C < \gamma^*$, there exist a strictly positive t^* such that $dC(t)/dt < 0$ for all $0 < t < t^*$, $dC(t)/dt = 0$ for $t = t^*$, and $dC(t)/dt > 0$ for all $t > t^*$. Indeed, we can easily get the expression of t^*:

$$t^* = \frac{\ln\left[\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\right]}{Z}.$$

The proof is provided in Appendix D.

We summarize the monotonicity properties of $C(t)$ in the following proposition:

Proposition 10.

(1) *If $S(0)/K(0) \geq (S(0)/K(0))^*$, then $C(t)$ increases monotonically for all t.*

(2) *If $S(0)/K(0) < (S(0)/K(0))^*$, then*

- **(2.1)** *if $\gamma_\theta + \frac{1}{\alpha}\gamma_A \geq \gamma^*$ then $C(t)$ increases monotonically for all $t > 0$;*
- **(2.2)** *if $\gamma_\theta + \frac{1}{\alpha}\gamma_A < \gamma^*$ then $C(t)$ then $C(t)$ decreases monotonically for all $0 < t < t^*$, and increases monotonically for all $t > t^*$.*

The economic interpretation of Propositions 9 and 10 is the following. Let us consider an economy where the proportion of non-renewable energy resources to endowment of physical capital is high enough, *i.e.*, $S(0)/K(0) > (S(0)/K(0))^*$ (see Boucekkine *et al.* (2008) for the importance of imbalance effects in global dynamics). Then, due to the relative abundance of energy resources, detrended consumption ($\tilde{C}(t)$) increases monotonically until its BGP level. However, if the endowment of non-renewable energy resources is relatively low (*i.e.*, $S(0)/K(0) < (S(0)/K(0))^*$), the economy can not keep its initial level of consumption since energy resources are required to build new equipment. Therefore, detrended consumption monotonically declines until the corresponding BGP level (see Proposition 9).[19] Let us study $C(t)$

(see Proposition 10). If the economy is relatively abundant in non-renewable energy resources (*i.e.*, $S(0)/K(0) \geq (S(0)/K(0))^*$) then $C(t)$ increases monotonically. However, under relative scarcity of non-renewable energy resources $C(t)$ does not necessarily increase for all $t > 0$. Indeed, an economy relatively poor in energy resources increases (monotonically) $C(t)$ if technical progress (in particular, energy-saving) grows at a high enough rate (*i.e.*, $\gamma_\theta + \frac{1}{\alpha}\gamma_A \geq \gamma^*$). Nevertheless, if despite its lack of energy resources the economy keeps technical progress at a sufficiently low growth rate (*i.e.*, $\gamma_\theta + \frac{1}{\alpha}\gamma_A < \gamma^*$), then consumption declines monotonically during t^* periods. After that, the "accumulated" technical progress (notice that (exogenous) technical progress increases with time) overcomes the scarcity of non-renewable energy resources and consumption increases monotonically for all $t > t^*$. Moreover, it is easy to prove that

$$\frac{dt^*}{d\rho} = \frac{1}{\gamma_C(Z - \alpha\gamma_C)}.$$

Thus, from Proposition 2 and taking the definition of Z (see Appendix A, equation (A.8)), we conclude that $dt^*/d\rho > 0$. This result implies that the more impatient the household is (*i.e.,* the greater is ρ), the greater is the number of periods where consumption decreases (t^*).[20] Consequently, Proposition 10 predicts that economies relatively poor in energy resources will face a transition with declining consumption during some periods if the growth rate of technical progress (in particular, energy-saving) is not high enough. This kind of transitional dynamics of consumption was previously observed in several studies on non-renewable energy resources and technical progress (see for instance Jorgenson and Wilcoxen (1993) and Growiec and Schumacher (2008)). Nevertheless, due to the analytical complexity of the problem, their results were based on numerical simulations. However, our paper provides an analytical set-up for this outcome, where the relative scarcity of resources and the technical progress are involved. Moreover, we should notice this is a global dynamics result and, consistent with Krautkraemer (1998), is also related with the impatient degree of the economy. As it is clear, this non-monotonicity result is particularly relevant for developed economies such as the European Union (EU), where the relative endowment of non-renewable energy resources is low. Indeed, as the IEA (2006) shows, the oil supply of the EU in 2005 was 4.8 billion barrels per day (5.74% of total world supply), with an estimated average annual growth rate of -4.5% for the period 2005-2030. However, the primary oil demand of the EU in 2005 (13.5 billion barrels per day (16.15% of total world demand), with an estimated average annual growth rate of 0.2% for the period 2005-2030) was almost three times the oil supply of the EU in 2005.[21] Actually, this shortage of energy resources has recently driven the EU to show an explicit interest in improving energy-efficiency by means of energy-saving technologies. Indeed, the European Commission (COM (2005)) estimated that the EU could have saved, at least, 20% of its energy consumption in 2005 by adopting energy-saving technologies. Moreover, as Boucekkine and Pommeret (2004) pointed out, the energy-saving technological progress has been significant in the last two decades (see for instance Newell *et al.* (1999) and Kuper and Soest (2003)). Indeed, the energy intensity of OCDE countries for the period 1990-2004 has been reduced by 1.6% per year (IEA (2006)). Therefore, our theoretical result goes in the same direction of promoting energy-efficiency policies such as the adoption of energy-saving technologies.

To conclude the monotonicity study of consumption, we compare our results with Dasgupta and Heal (1974). In a model with identical technology for both sectors, physical capital accumulation and consumption, these authors observed that, the optimal trajectories of consumption are not necessarily monotonic. However, our paper introduces five novelties with respect to their study. First, as we already pointed out, our study is global (*i.e.*, whatever the initial conditions), while Dasgupta and Heal (1974) consider local dynamics. Second, our sufficient condition for non-monotonicity involves the proportion of initial endowment of non-renewable resources ($S(0)$) to the initial endowment of physical capital ($K(0)$), *i.e.*, the ratio $S(0)/K(0)$. However, what matters for Dasgupta and Heal (1974)'s non-monotonicity result is the size of $S(0)$ and $K(0)$, *i.e.*, not necessarily the size of the ratio $S(0)/K(0)$. Third, our sufficient condition entails both the ratio $S(0)/K(0)$ and the growth rate of technical progress. This contrasts with Dasgupta and Heal (1974) since their sufficient condition does not involve technical progress. Fourth, Dasgupta and Heal (1974) show that if the size of both $S(0)$ and $K(0)$ is large enough then consumption initially rises and, after some time, it will decrease. However, our model raises the possibility of an opposite behaviour of consumption. Indeed, if both $S(0)/K(0)$ and the growth rate of technical progress are low enough then consumption depicts a U-shape. Moreover, we analytically prove that the more impatient the household, the greater is the number of periods where consumption decreases. Finally, as we will see in the next section, the fifth novelty is that our model shows that resource extraction is not necessarily monotonic. This contrasts with Dasgupta and Heal (1974) where extraction decreases monotonically.

5.3 Monotonicity of extraction of non-renewable energy resources

From Equation (15), we get the detrended $R(t)$:

$$\tilde{R}(t) =$$

$$\cdot \left(\frac{Z}{\varphi Z + \alpha \overline{A} \theta^{\alpha}} \right)^{\frac{1}{\alpha}} \frac{\overline{A} K(0)}{{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A} \theta^{\alpha}} \right)} \left[{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \right) - \alpha \frac{Z + r}{Z} \right] \tag{22}$$

Taking logs and differentiating with respect to t, it yields

$$\frac{d\tilde{R}(t)}{dt} = \left[\tilde{R}(t) \frac{Z + \rho}{2Z + \rho} \frac{Z^2}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \frac{{}_2F_1 \left(a, b, c; -\frac{\varphi Z}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \right)}{{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \right) - \alpha \frac{Z+\rho}{Z}} \right] \varphi, \tag{23}$$

where

$$a = 2; b = 1 + \frac{Z + \rho}{Z}; c = b + 1.$$

Following Corollary 7, we observe that the terms between square brackets in equation (23) are positive. Thus, the sign of $d\tilde{R}(t)/dt$ is determined by the sigh of φ. Hence, we establish the following proposition:

Proposition 11.

(1) *If $S(0)/K(0) > (S(0)/K(0))^*$, then $\tilde{R}(t)$ decreases monotonically for all t.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*[22]

$$\tilde{R}(t) = \overline{A} K(0) \left(\frac{Z}{\alpha \overline{A} \theta^{\alpha}} \right)^{\frac{1}{\alpha}} \left(1 - \alpha \frac{Z + \rho}{Z} \right).$$

(3) *If $S(0)/K(0) < (S(0)/K(0))^*$, then $\tilde{R}(t)$ increases monotonically for all t.*

Since $R(t) = \tilde{R}(t) \exp(-\rho t)$, then $dR(t)/dt = \exp(-\rho t)(d\tilde{R}(t)/dt - \rho \tilde{R}(t))$. Therefore, applying equation (23), we obtain

$$\frac{dR(t)}{dt} = \tilde{R}(t) \exp(-\rho t)$$

$$\cdot \left\{ \left[\frac{Z + \rho}{2Z + \rho} \frac{Z^2}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \frac{{}_2F_1 \left(a, b, c; -\frac{\varphi Z}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \right)}{{}_2F_1 \left(-\frac{\varphi Z}{\alpha \overline{A} \theta^{\alpha} \exp(Zt)} \right) - \alpha \frac{Z+\rho}{Z}} \right] \varphi - \rho \right\}. \tag{24}$$

As the terms between square brackets are positive, we establish the monotonicity properties of $R(t)$:

Proposition 12.

(1) *If* $S(0)/K(0) \geq (S(0)/K(0))^*$*, then* $R(t)$ *decreases monotonically for all* t*.*

(2) *If* $S(0)/K(0) < (S(0)/K(0))^*$*, then* $R(t)$ *is not necessarily monotonic.*

As we observed in the previous section, this result contrasts with Dasgupta and Heal (1974). Indeed, these authors obtained that $R(t)$ decreases monotonically for all t. However, since we assume that physical capital accumulation sector is relatively more energy intensive than consumption, our model allows for a non-monotonic behavior of $R(t)$ if the economy is relatively poor in energy resources.[23]

6 Concluding remarks

In this paper we studied the implications of assuming that physical capital accumulation is relatively more energy-intensive than consumption, focusing on the analysis of the BGP. By means of Gaussian hypergeometric functions we provided a closed-form representation of the optimal solution paths of our variables in levels, whatever the initial conditions (*i.e.*, global dynamics). We also proved that there is asymptotic convergence to the BGP. Moreover, we explicitly studied the monotonicity of the optimal trajectories of the variables in levels, which is typically unfeasible in the standard approach. We found that, in general, the optimal trajectories are non-monotonic. Dasgupta and Heal (1974) already pointed out this result (local dynamics) for the optimal consumption in a model with identical technology for both physical capital accumulation and consumption. However, our paper introduced five novelties with respect to their study (namely, global dynamics, the importance of the size of the ratio $S(0)/K(0)$, the role of the technical progress, U-shape behaviour of consumption, and the non-monotonicity of resource extraction), which are direct consequence of assuming both a physical capital accumulation sector that is relatively more energy intensive than consumption, and growing technical progress (in particular, energy-saving). Moreover, it is important to note that this new dynamical transition undoubtedly has important welfare implications, in particular with regard to the adoption of energy-saving technologies. However, since this is not the scope of the present paper, we leave the welfare analysis for further research.

Our model has several limitations. Among of them, we point out the following two. First, as Solow (1974a), we assumed unlimited (exogenous) technical progress. In regard to our disembodied technical progress ($A(t)$), this assumption captures the idea of unlimited growth of knowledge. However, unlimited energy-saving technical progress ($\theta(t)$) could arise physical incompatibility problems related to the thermodynamic laws (see Krautkraemer (1998)). In the literature of exogenous energy-saving technical progress, the assumption of unlimited growth of energy efficiency is widely applied (see for instance Boucekkine and Pommeret (2004), Azomahou *et al.* (2006), and Pérez-Barahona and Zou (2006a,b)). Indeed, these authors assume that the limit of improving energy efficiency is very far away. In our model, we take the same point of view. A second limitation of our analysis is that we assumed exogenous technical progress. The reason of making the assumption is that the aim of this paper is to study the conditions under which technical progress can offset the trade-off between economic growth and the usage of non-renewable energy resources, and not to analyze how the economy can achieve these conditions. However, a very interesting extension to our model is to assume technical progress (in particular, energy-saving) as an endogenous variable. Actually, as we observed in Proposition 2, our BGP growth rates are not affected by the stock of non-renewable energy resources. However, the scarcity of energy resources can induce new investments in energy-saving technologies (see for instance, Newell *et al.* (1999), Buonanno *et al.* (2003), Carraro *et al.* (2003) and Boucekkine and Pommeret (2004)), establishing an effect of the stock of non-renewable energy resource on the BGP growth rates. Indeed, one possibility to endogenize energy-saving technical progress could be by means of an R&D sector (for energy-saving technologies) following Aghion and Howitt (1992) and Grossman and Helpman (1991a,b).

Appendix A

Following the description presented in Section 4, this appendix solves recursively the dynamical system (6)-(10). Since our paper focuses on the BGP, it is useful to detrend the dynamical system.[24] Therefore, let us define the following change of variable $\tilde{x}(t) = x(t)/\exp(\gamma_x t)$. Then, the transformed (detrended) variables will be constant along the BGP (if it exists). Applying the previous change of variable, we get the following detrended dynamical system with five equations and five unknowns $(\tilde{K}, \tilde{C}, \tilde{R}, \tilde{I}, S)$:

$$\dot{\tilde{K}}(t) + \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right)\tilde{K}(t) = \overline{\theta}^\alpha \tilde{R}(t)^\alpha \tilde{I}(t)^{1-\alpha}, \tag{A.1}$$

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = (1-\alpha)\overline{A\theta}^\alpha \left(\frac{\tilde{R}(t)}{\tilde{I}(t)}\right)^\alpha - \alpha\left(\frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)} - \frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)}\right) - \gamma_\theta - \frac{1-\alpha}{\alpha}\gamma_A, \tag{A.2}$$

$$(1-\alpha)\overline{A\theta}^\alpha \left(\frac{\tilde{R}(t)}{\tilde{I}(t)}\right)^\alpha = (1-\alpha)\left(\frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)} - \frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)}\right) + \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A, \tag{A.3}$$

$$\tilde{C}(t) + \tilde{I}(t) = \overline{A}\tilde{K}(t), \tag{A.4}$$

$$\dot{S}(t) = \tilde{R}(t)\exp(-\rho t). \tag{A.5}$$

Ratio $X(t)$

Let us define the ratio $X(t) = \tilde{R}(t)/\tilde{I}(t)$. Taking logs and differentiating with respect to t, we get

$$\frac{\dot{X}(t)}{X(t)} = \frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)} - \frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)}. \tag{A.6}$$

Bernoulli's equation

Applying equation (A.6) into equation (A.3), it yields

$$\dot{X}(t) = \frac{\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A}{1-\alpha} X(t) + \overline{A\theta}^{\alpha} X(t)^{1+\alpha}. \tag{A.7}$$

Notice that equation (A.7) is a Bernoulli's equation. Indeed, following Sydsæter *et al.* (1999), a Bernoulli's equation is defined as

$$\dot{x}(t) = q(t)x(t) + r(t)x(t)^n,$$

which has the following solution:

$$x(t) = \exp\left(\frac{p(t)}{1-n}\right)\left(\varphi + (1-n)\int r(t)\exp(-p(t))dt\right)^{\frac{1}{1-n}},$$

where $p(t) = (1-n)\int q(t)dt$, φ is a constant, and $n \neq 1$.

Therefore, the solution of the Bernouilli's equation (A.7) is

$$X(t) = \exp\left(\frac{1}{\alpha}Zt\right)\left(\varphi + \frac{\alpha\overline{A\theta}^{\alpha}}{Z}\exp(Zt)\right)^{-\frac{1}{\alpha}}, \tag{A.8}$$

where $Z = \frac{\alpha}{1-\alpha}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right)$, and φ is a constant which should be calculated.

Consumption

Applying equations (A.6) and (A.8) into equation (A.2), we find that

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = \frac{\overline{A\theta}^{\alpha}\exp(Zt)}{\varphi + \alpha\overline{A\theta}^{\alpha}\exp(Zt)} - \frac{Z}{\alpha}, \tag{A.9}$$

which is a linear first-order differential equation.

A linear first-order differential equation is defined as

$$\dot{x}(t) + a(t)x(t) = b(t),$$

where the corresponding solution is given by the expression (Sydsæter *et al.* (1999))

$$x(t) = x(0)\exp\left(-\int_0^t a(\xi)d\xi\right) + \int_0^t b(\tau)\exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau.$$

Applying this result into equation (A.9), it yields

$$\tilde{C}(t) = \tilde{C}(0)\exp\left[\frac{1}{\alpha}\left(\ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Zt)) - \ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}) - Zt\right)\right], \tag{A.10}$$

where $\tilde{C}(0)(= C(0))$ is the initial condition for consumption, which should be determined.

Physical capital

Rewriting equation (A.1), we get

$$\dot{\tilde{K}}(t) + \left(\frac{1-\alpha}{\alpha}Z - \rho - \overline{A\theta}^{\alpha} X(t)^{\alpha}\right)\tilde{K}(t) = -\overline{\theta}^{\alpha} X(t)^{\alpha}\tilde{C}(t) \qquad (A.11)$$

Since we know the analytical solution for $X(t)$ (equation (A.8)) and $\tilde{C}(t)$ (equation (A.10)), $\tilde{K}(t)$ is the solution of the linear first-order differential equation (A.11). Therefore, by applying the previous result of the solution of a linear first-order differential equation, we find that

$$\tilde{K}(t) = \exp\left[-\frac{1}{\alpha}\ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right]$$

$$\cdot \exp\left\{\frac{1}{\alpha}\left[\ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Zt)) - ((1-\alpha)Z - \alpha\rho)t\right]\right\}$$

$$\cdot\left\{K(0) + \frac{Z}{Z+\rho}\frac{\tilde{C}(0)}{\alpha\overline{A}}\left[\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)}{\exp((Z+\rho)t)} - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\right]\right\}, \qquad (A.12)$$

where

$${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1 + \frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right),$$

$${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1 + \frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right),$$

and $K(0)(=\tilde{K}(0))$ is the (given) initial condition for capital.

Notice that equation (A.12) involves Special Functions (${}_2F_1$). More precisely, we use a family of Special Functions called Gaussian hypergeometric functions, which are defined as

$${}_pF_q(a;b;z) = \sum_{n=0}^{\infty}\frac{(a_1)_n \times (a_2)_n \times \ldots (a_p)_n}{(b_1)_n \times (b_2)_n \times \ldots (b_q)_n}\frac{z^n}{n!},$$

where $a = (a_1, a_2, \ldots, a_p)$, $b = (b_1, b_2, \ldots, b_q)$, and $(x)_n$ is the Pochhammer symbol, defined by:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)},$$

where $\Gamma(x)$ is the *Gamma function*.

To prove equation (12) we follow Abramowitz and Stegun (1970) and Boucekkine and Ruiz-Tamarit (2008).[25] The proof is available upon request (see Appendix E, proof of equation (A.12)).

Investment

Since we have the solution for $\tilde{C}(t)$(equation (A.10)) and $\tilde{K}(t)$(equation (A.12)), equation (A.4) yields

$$\tilde{I}(t) = \exp\left\{\frac{1}{\alpha}\left[\ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(zt)) - \ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}) - Zt\right]\right\}$$

$$\cdot\left\{\overline{A}\exp((Z+\rho)t)\zeta(t) - \tilde{C}(0)\right\}, \qquad (A.13)$$

where

$$\zeta(t) =$$

$$\left\{K(0) + \frac{Z}{Z+\rho}\frac{\tilde{C}(0)}{\alpha\overline{A}}\left[\frac{1}{\exp((Z+\rho)t)}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right) - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\right]\right\}.$$

Resources extraction

Since $X(t)$ is defined as $X(t) = \tilde{R}(t)/\tilde{I}(t)$, taking the solution for $X(t)$ (equation (A.8)) and $\tilde{I}(t)$ (equation (A.13)), we find the solution for the energy flow $\tilde{R}(t)$:

$$\tilde{R}(t) = \exp\left[-\frac{1}{\alpha}\ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right] Z^{\frac{1}{\alpha}} \exp\left(\frac{1}{\alpha}Zt\right)$$

$$\cdot \left\{\overline{A}\exp\left\{\left(\frac{1-\alpha}{\alpha}Z - \rho\right)t\right\}\zeta(t) - \tilde{C}(0)\exp\left(-\frac{1}{\alpha}Zt\right)\right\}. \quad (A.14)$$

Resources stock

Since we have the solution for $\tilde{R}(t)$ (equation (A.14)), we find the solution for the stock of non-renewable energy resources, $S(t)$, by solving equation (A.5), which is a linear first-order differential equation. The solution of this equation is

$$S(t) = S(0) + \int_0^t \tilde{R}(\tau)\exp(-\rho\tau)d\tau, \quad (A.15)$$

where $S(0)$ is given. The corresponding integral is equal to

$$\int_0^t \tilde{R}(\tau)\exp(-\rho\tau)d\tau = [1] + [2] + [3] + [4],$$

where

$$[1] = \frac{(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A}K(0)}{2(\frac{1}{\alpha}Z - \rho) - Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z - \rho\right) - Z\right)t\right] - 1\right\},$$

$$[2] = A_2 \frac{1}{Z+\Omega}\frac{Z}{Z+\rho}$$

$$\cdot \left[{}_3F_2\left(a; b; -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) - \exp\left(-(Z+\Omega)\frac{Z+\rho}{Z}t\right){}_3F_2\left(a; b; -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)\right],$$

$$[3] = -A_2\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{2(\frac{1}{\alpha}Z - \rho) - Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z - \rho\right) - Z\right)t\right] - 1\right\},$$

$$[4] = \frac{(\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho}\left(\exp(-\rho t) - 1\right),$$

with

$$A_2 = (\varphi Z + \alpha\overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}}\frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha},$$

$$\Omega = \left(2\rho - \frac{2-\alpha}{\alpha}Z\right)\left(\frac{Z}{Z+\rho}\right),$$

$$a = \left(1, \frac{Z+\rho}{Z}, \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right),$$

$$b = \left(1 + \frac{Z+\rho}{Z}, 1 + \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right),$$

and

$$\left|-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right| < 1.$$

The proof is available upon request (see Appendix E, proof of equation (A.15)).

Computation of the constants φ and $C(0)$

Equations (A.10), and (A.12)-(A.15) are the analytical solutions for, respectively, $\tilde{C}(t)$, $\tilde{K}(t)$, $\tilde{I}(t)$, $\tilde{R}(t)$, and $S(t)$. Moreover, by retrieving the change of variable (notice that γ_A and γ_θ are exogenous parameters) we recover the original variables involved in the dynamical system (6)-(10). However, we still need to determine the constants φ and $C(0)$ to conclude our calculations. To do this, we use the two following transversality conditions (TC) corresponding to each state variable of our model:

$$\lim_{t\to\infty} \psi(t)K(t) = 0, \tag{A.16}$$

$$\lim_{t\to\infty} \lambda(t)S(t) = 0, \tag{A.17}$$

where $\psi(t)$ and $\lambda(t)$ are the Lagrangian multipliers. In addition, the TC also imply restrictions on the parameters, which allow us to complete the characterization of the optimal trajectories.

TC for the stock of physical capital ($K(t)$)

By solving the optimal control problem presented in Section 2.4, it is easy to obtain the shadow price of physical capital, $\psi(t)$:

$$\psi(t) = \frac{(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{\alpha}}{(1-\alpha)\overline{\theta}^{\alpha}C(0)}[\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)]^{-\frac{1-\alpha}{\alpha}}. \tag{A.18}$$

Taking equation (A.12), we get[26]

$$\psi(t)K(t) = \frac{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}{(1-\alpha)\overline{\theta}^{\alpha}C(0)}$$

$$\cdot\left\{K(0) + \frac{Z}{Z+\rho}\frac{C(0)}{\alpha\overline{A}}\left[\exp(-(Z+\rho)t)\,{}_2F_1\left(\frac{-\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right) - {}_2F_1\left(\frac{-\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)\right]\right\}.$$

Then, the TC for $K(t)$ (equation (A.16)) implies that

$$C(0) = \frac{\alpha\overline{A}}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}\frac{Z+\rho}{Z}K(0), \tag{A.19}$$

where $K(0)$ is given.

Equation (A.19) establishes a first relationship between φ and $C(0)$. However, we need an additional expression between these two constants to complete the job. In the following, we will obtain this second relationship from the TC for the non-renewable energy resources (equation (A.17)).

TC for the stock of non-renewable energy resource ($S(t)$)

Since it is not optimal to completely deplete the stock of non-renewable energy resource in a finite t,[27] we can proof that $\lambda(t) = \lambda$ for all t. Therefore, equation (A.17) implies that the stock of non-renewable energy resources should be depleted in the infinite, *i.e.*,

$$\lim_{t\to\infty} S(t) = 0. \tag{A.20}$$

Taking equation (A.19) into equation (A.15), we get that

$$S(t) = S(0) - \frac{(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}}Z^{\frac{1}{\alpha}}C(0)}{\rho}(\exp(-\rho t) - 1) - A_2\frac{1}{Z+\Omega}\frac{Z}{Z+\rho}$$

$$\cdot \left[{}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right) - \exp\left(-(Z+\Omega)\frac{Z+\rho}{Z}t\right) {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right)\right], \tag{A.21}$$

where

$$ {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right) = {}_3F_2\left(a; b; -\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right),$$

$$ {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right) = {}_3F_2\left(a; b; -\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right).$$

From equation (A.21), if the term $(\Omega+Z)\frac{Z+\rho}{Z}$ is negative,[28] then $S(t)$ will infinitely increase. Hence, $(\Omega+Z)\frac{Z+\rho}{Z}$ should be greater than zero in order to avoid explosive solutions. Since ρ and Z are positive, we have to ensure that $\Omega + Z$ is positive too.[29] Then, from the definition of Ω, we can establish that any particular solution to the dynamical system (6)-(10) has to satisfy the condition $\rho > \frac{2}{3}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right)$. Moreover, as we showed in Section 2, the growth rate of technical progress should be high enough if BGP exists. Therefore, we can consider both conditions in the following proposition:

Proposition A.1. *Any particular BGP solution to the dynamical system (6)-(10) has to satisfy the following condition:*

$$\frac{2}{3}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right) < \rho < \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right). \tag{A.22}$$

Finally, we can establish a condition on the parameters to deplete the stock of non-renewable energy resources in the infinite. From equation (A.21) and condition (A.22), it is clear that

$$\lim_{t\to\infty} S(t) =$$

$$S(0) + \frac{C(0)}{\rho}\left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}} - A_2 \frac{1}{Z+\Omega}\frac{Z}{\Omega+\rho} {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right). \tag{A.23}$$

From equation (A.20), $\lim_{t\to\infty} S(t) = 0$. Then, we obtain the following condition by substituting A_2 (see equation (A.15)) into equation (A.23) and equalizing to zero:

$$S(0) =$$

$$C(0)\left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}} \left[\left(\frac{Z}{Z+\rho}\right)^2 \frac{1}{\alpha}\frac{1}{Z+\Omega} {}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right) - \frac{1}{\rho}\right], \tag{A.24}$$

where $S(0)$ is given.

Equation (A.24) is the second relationship between φ and $C(0)$. Thus, equations (A.19) and (A.24) constitute a non-linear system of two equations and two unknowns. Taking equation (A.19) into equation (A.24), we reduce the previous non-linear system of two equations ((A.19) and (A.24)) and two unknowns (φ and $C(0)$) to the following single equation for φ, where $C(0)$ is directly found (equation (A.19)) once φ is determined:

$$\frac{S(0)}{K(0)} = \alpha \overline{A}\left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot \left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)} - \frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}\right], \tag{A.25}$$

where $K(0)$ and $S(0)$ are given.

Since equation (A.15) requires $\left|-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right| < 1$, we only consider parameterizations with a solution for equation (A.25) such that[30]

$$\varphi \in \left(-\frac{\alpha \overline{A}\theta^{\alpha}}{Z}, \frac{\alpha \overline{A}\theta^{\alpha}}{Z}\right). \tag{A.26}$$

Appendix B

Optimal solution paths for the ratio $X(t)$

From equation (A.8), rearranging terms, we get the following proposition for the optimal solution path for $X(t)$:

Proposition B.1. *Under conditions (A.22) and (A.26), if equation (A.25) admits a unique solution, then*

(i) *there exists a unique path for $X(t)$, starting from $X(0) = \left(\frac{Z}{\varphi Z+\alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$, such that*

$$X(t) = \left(\frac{Z}{\varphi Z \exp(-Zt) + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}; \tag{B.1}$$

(ii) *this equilibrium path shows transitional dynamics, approaching asymptotically to the constant*

$$X_{BGP} = \left(\frac{Z}{\alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}.$$

Moreover, it is easy to see that condition (A.26) implies that the starting point, the transitional dynamics, and the BGP of $X(t)$ are always strictly positive:

Corollary B.1. *Under conditions (A.22) and (A.26), the optimal solution path for $X(t)$ is strictly positive.*

Appendix C

Let us define

$$F\left(\frac{S(0)}{K(0)}\right) = \frac{S(0)}{K(0)} - \alpha\overline{A}\left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)} - \frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)}\right]. \tag{C.1}$$

Condition (A.25) implies

$$F\left(\frac{S(0)}{K(0)}\right) = 0.$$

Then, applying the implicit function theorem, we get

$$\frac{\partial \varphi}{\partial(S(0)/K(0))} = -\frac{\partial F/\partial(S(0)/K(0))}{\partial F/\partial \varphi}. \tag{C.2}$$

From (C.1), $\partial F/\partial(S(0)/K(0)) = 1$. Moreover, $\partial F/\partial \varphi = -\partial g(\varphi)/\partial \varphi$, where

$$g(\varphi) = \alpha \overline{A}\left(\frac{Z}{\varphi Z + \alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}-\frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}\right]. \qquad (C.3)$$

Applying the following property (Luke (1975), Vol.1, page 111 (Property 38))

$$ {}_3F_2\left\{(a,b,c);(d+1,c+1);Z\right\}=\frac{c}{c-d}{}_2F_1\left(a,b,d+1;Z\right)$$

$$-\frac{d}{c-d}{}_3F_2\left\{(a,b,c);(d,c+1);Z\right\},$$

we obtain that

$$\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}=\frac{Z+\Omega}{\Omega}-\frac{Z}{\Omega}\frac{{}_2F_1\left(a',b',c',-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\right)},$$

where

$$a'=1;b'=\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z};c'=1+b'.$$

Thus, we can rewrite $g(\varphi)$ with ${}_2F_1$ functions. Applying the formula for the derivative of ${}_2F_1$ (see Abramowitz and Stegun, 15.2.1, page 557), we get $\partial g(\varphi)/\partial\varphi$. Rearranging terms and taking Corollary 7, we obtain that $\partial g(\varphi)/\partial\varphi<0$. Therefore, from (C.2), we conclude that

$$\frac{\partial\varphi}{\partial(S(0)/K(0))}<0$$

■

Appendix D

Let us consider the case

$$\gamma_C<\frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z+\alpha\overline{A}\theta^\alpha}.$$

Taking

$$\gamma_C=\frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z+\alpha\overline{A}\theta^\alpha\exp(Zt^*)}, \qquad (D.1)$$

we obtain t^*. Therefore, from equation (D.1) we get that

$$\exp(Zt^*)=\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right). \qquad (D.2)$$

Since, under Corollary 7, the term $Z-\alpha\gamma_C$ is strictly positive, we can apply logs in equation (D.2). Then,

$$t^*=\frac{\ln\left[\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\right]}{Z}.$$

If the term between square brackets is >1, then $t^*>0$. We can prove this by contradiction. Let us assume that

$$\frac{\varphi Z}{\alpha\overline{A}\theta^\alpha}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\leq 1.$$

Then, it easy to prove that

$$\gamma_C\geq\frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z+\alpha\overline{A}\theta^\alpha},$$

which contradicts our initial statement. Then $t^*>0$ ■

Appendix E

a) Proof of equation (A.12)

Equation (A.11) can be rewritten as a standard linear first-order differential equation:

$$\dot{\tilde{K}}(t) + a(t)\tilde{K}(t) = b(t), \tag{a.1}$$

where

$$a(t) = \frac{1-\alpha}{\alpha}Z - \rho - \overline{A}\overline{\theta}^{\alpha}X(t)^{\alpha},$$
$$b(t) = -\overline{\theta}^{\alpha}X(t)^{\alpha}\tilde{C}(t),$$

and $K(0)(=\tilde{K}(0))$ is given.

Since the previous steps provide the solutions for $X(t)$ (equation (A.8)) and $\tilde{C}(t)$ (equation (A.10)), we can solve the differential equation (a.1) by applying the following result (Sydsæter *et al.* (1999)):

$$\tilde{K}(t) = \tilde{K}(0)\exp\left(-\int_0^t a(\xi)d\xi\right) + \int_0^t b(\tau)\exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau. \tag{a.2}$$

Taking equations (A.8) and (A.10), we obtain that

$$a(t) = \frac{1-\alpha}{\alpha}Z - \rho - \frac{Z\overline{A}\overline{\theta}^{\alpha}}{\frac{\varphi Z}{\exp(Zt)} + \alpha\overline{A}\overline{\theta}^{\alpha}}. \tag{a.3}$$

After some calculations, we find

$$\int a(\xi)d\xi = \left(\frac{1-\alpha}{\alpha}Z - \rho\right)\xi + \frac{1}{\alpha}\ln\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Z\xi)}\right). \tag{a.4}$$

Evaluating the integral (a.4) in t and 0, and after rearranging terms, we obtain the first part of equation (a.2):

$$\tilde{K}(0)\exp\left(-\int_0^t a(\xi)d\xi\right) = \tilde{K}(0)\exp\left(-\frac{1}{\alpha}\ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})\right)$$
$$\cdot\exp\left(\frac{1}{\alpha}(\ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}) - ((1-\alpha)Z - \alpha\rho)t)\right).$$

In order to calculate the second part of equation (a.2), we first obtain

$$b(\tau)\exp\left(-\int_\tau^t a(\xi)d\xi\right) =$$
$$-Z\overline{\theta}^{\alpha}C(0)\exp\left\{\frac{1}{\alpha}[\ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)) - \ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})]\right\}$$
$$\cdot\exp\left\{\left(-\frac{1-\alpha}{\alpha}Z + \rho\right)t\right\}\frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Z\tau)}, \tag{a.5}$$

by evaluating the integral (a.4) in τ and t. Then,

$$\int_0^t b(\tau)\exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau =$$
$$-Z\overline{\theta}^{\alpha}C(0)\exp\left\{\frac{1}{\alpha}[\ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)) - \ln(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})]\right\}$$
$$\cdot\exp\left\{\left(-\frac{1-\alpha}{\alpha}Z + \rho\right)t\right\}$$

$$\cdot \int_0^t \frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Z\tau)} d\tau. \tag{a.6}$$

In order to calculate equation (a.6), we need to obtain

$$\int_0^t \frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A\theta}^{\alpha}\exp(Z\tau)} d\tau. \tag{a.7}$$

To do this, we follow Boucekkine and Ruiz-Tamarit (2008). Equation (a.7) is equal to

$$\int_0^t \Big(\varphi Z \exp(\rho\tau) + \alpha\overline{A\theta}^{\alpha}\exp((Z+\rho)\tau)\Big)^{-1} d\tau.$$

Taking the following change of variable $V = \exp(-(\rho+Z)\tau)$,

$$\int_0^t \Big(\varphi Z \exp(\rho\tau) + \alpha\overline{A\theta}^{\alpha}\exp((Z+\rho)\tau)\Big)^{-1} d\tau =$$

$$\int_1^{\exp(-(\rho+Z)t)} \left(1 + \frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}} V^{\frac{Z}{Z+\rho}}\right)^{-1} dV. \tag{a.8}$$

Recalling $y = \exp(-(\rho+Z)t)$, $A_0 = \frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}$, and $\beta = -1$, equation (a.8) can be rewritten as

$$\int_1^y \left(1 + A_0 V^{\frac{Z}{Z+\rho}}\right)^{\beta} dV. \tag{a.9}$$

Applying the binomial theorem:

$$(1+ax^b)^c = \sum_{n=0}^{\infty} (-c)_n \frac{(-ax^b)^n}{n!},$$

equation (a.9) equals

$$\sum_{n=0}^{\infty} \frac{(-\beta)_n}{n!}(-A_0)^n \int_1^y V^{\frac{Z}{Z+\rho}n} dV,$$

which is equal to

$$\sum_{n=0}^{\infty} (-\beta)_n \frac{-(A_0)^n}{n!} \frac{1}{\frac{Z}{Z+\rho}n+1}\left(y^{\frac{Z}{Z+\rho}n} y - 1\right). \tag{a.10}$$

Applying the property $\frac{X}{X+n} = \frac{(X)_n}{(1+X)_n}$, equation (a.10) equals

$$y \sum_{n=0}^{\infty} \frac{(a)_n(-\beta)_n}{(1+a)_n} \frac{(-A_0 y^{\frac{Z}{Z+\rho}})^n}{n!}$$

$$-\sum_{n=0}^{\infty} \frac{(a)_n(-\beta)_n}{(1+a)_n} \frac{(-A_0)^n}{n!}. \tag{a.11}$$

Taking the definition of the Gaussian hypergeometric function

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

we find that equation (a.8) equals

$$\exp(-(\rho+Z)t)\,{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right) - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right), \tag{a.12}$$

where

$$ {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right), $$

$$ {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right) $$

Hence, retrieving the change of variables,

$$ \int_0^t \frac{\exp(-\rho\tau)}{\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha}\exp(Z\tau)}d\tau = $$

$$ \frac{1}{\alpha\overline{A}\overline{\theta}^{\alpha}(Z+\rho)}\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right) - \frac{1}{\exp[(\rho+Z)t]}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right)\right]. \tag{a.13} $$

Finally, taking together equations (a.3), (a.5), and (a.13), and rearranging terms, we easily obtain equation (A.12) ■

b) Proof of equation (A.15)

Equation (A.14) can be rewritten as follows:

$$ \tilde{R}(t) = [a] + [b] + [c] + [d], $$

where

$$ [a] = (\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A}K(0)\exp\left\{\left(\frac{2-\alpha}{\alpha}Z - \rho\right)t\right\}, $$

$$ [b] = (\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}\exp(Zt)}\right)\exp\left\{2\left(\frac{1-\alpha}{\alpha}Z - \rho\right)t\right\}, $$

$$ [c] = -(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)\exp\left\{\left(\frac{2-\alpha}{\alpha}Z - r\right)t\right\}, $$

$$ [d] = -(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0). $$

Then,

$$ \int_0^t \tilde{R}(\tau)\exp(-\rho\tau)d\tau = \int_0^t [a]\exp(-\rho\tau)d\tau + \int_0^t [b]\exp(-\rho\tau)d\tau $$
$$ + \int_0^t [c]\exp(-\rho\tau)d\tau + \int_0^t [d]\exp(-\rho\tau)d\tau. $$

It is easy to prove that

$$ \int_0^t [a]\exp(-\rho\tau)d\tau = $$

$$ \frac{(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}}\overline{A}K(0)}{2(\frac{1}{\alpha}Z - \rho) - Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z - \rho\right) - Z\right)t\right] - 1\right\} = [1] $$

$$ \int_0^t [c]\exp(-\rho\tau)d\tau = -A_2\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}\right)}{2(\frac{1}{\alpha}Z - \rho) - Z}\left\{\exp\left[\left(2\left(\frac{1}{\alpha}Z - \rho\right) - Z\right)t\right] - 1\right\} = [3] $$

$$ \int_0^t [d]\exp(-\rho\tau)d\tau = \frac{(\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho}(\exp(-\rho t) - 1) = [4], $$

where

$$ A_2 = (\varphi Z + \alpha\overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}}\frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho}\frac{C(0)}{\alpha}. $$

The integral [b] involves ${}_3F_2$ Hypergeometric functions. Indeed, applying the Euler integral representation (Abramowitz and Stegun (1970), page 558, formula 15.3.1), we obtain that

$$\int_0^t [b]\exp(-\rho\tau)d\tau = A_2\frac{Z+\rho}{Z}(\alpha\overline{A\theta}^{\alpha})^{\frac{Z+\rho}{Z}}$$

$$\cdot\int_0^1 (1-\xi)^{\frac{\rho}{Z}}\left(\int_0^t \left(\alpha\overline{A\theta}^{\alpha}\exp(Z\tau)+\varphi Z\xi\right)^{-\frac{Z+\rho}{Z}}\exp\left\{\left(\frac{2-\alpha}{\alpha}Z-2\rho\right)\tau\right\}d\tau\right)d\xi,$$

The integral $\int_0^t(\cdot)d\tau$ can be rewritten as

$$\int_0^t \left\{\varphi Z\xi\exp(R\tau)+\alpha\overline{A\theta}^{\alpha}\exp[(Z+\Omega)\tau]\right\}^{-\frac{Z+\rho}{Z}}d\tau, \qquad \text{(b.1)}$$

where

$$\Omega = -\left(\frac{2-\alpha}{\alpha}Z-2\rho\right)\frac{Z}{Z+\rho}.$$

Following a similar strategy as in equation (a.7), we can apply the change of variable $V=\exp\{(\Omega+\gamma)\tau\}$ to equation (b.1). Taking $\gamma=-(Z+\Omega)[(Z+\rho)/Z]-\Omega$, one can prove that equation (b.1) equals

$$-\left(\alpha\overline{A\theta}^{\alpha}\right)^{-\frac{Z+\rho}{Z}}\frac{Z}{Z+\rho}\frac{1}{Z+\Omega}$$

$$\cdot\left[\exp\left\{-(Z+\Omega)\frac{Z+\rho}{Z}t\right\}{}_2F_1\left(\hat{a},\hat{b},\hat{c};-\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)-{}_2F_1\left(\hat{a},\hat{b},\hat{c};-\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha}}\right)\right],$$

where

$$\hat{a}=\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z};\hat{b}=\frac{Z+\rho}{Z};\hat{c}=1+\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}.$$

Then,

$$\int_0^t [b]\exp(-\rho\tau)d\tau = A_2\frac{1}{Z+\Omega}$$

$$\cdot\left\{[INT1]-\exp\left(-(Z+\Omega)\frac{Z+\rho}{Z}t\right)[INT2]\right\}, \qquad \text{(b.2)}$$

where

$$[INT1]=\int_0^1(1-\xi)^{\frac{\rho}{Z}}{}_2F_1\left(\hat{a},\hat{b},\hat{c};-\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha}}\right)d\xi,$$

$$[INT2]=\int_0^1(1-\xi)^{\frac{\rho}{Z}}{}_2F_1\left(\hat{a},\hat{b},\hat{c};-\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha}\exp(Zt)}\right)d\xi.$$

Assuming that $\left|-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right|<1$, we can apply the following property (see Luke (1975), Vol.1, page 58, equation (10)):

$${}_{p+1}F_{q+1}\left\{(\beta,\alpha_p);(\beta+\sigma,\rho_p);Z\right\}=$$

$$\frac{\Gamma(\beta+\sigma)}{\Gamma(\beta)\Gamma(\sigma)}\int_0^1 t^{\beta-1}(1-t)^{\sigma-1}{}_{p+1}F_{q+1}\left\{(\alpha_p);(\rho_p\ Zt)\right\}dt$$

for $|Z|<1$, $Re(\beta)>0$, and $Re(\sigma)>0$.

In our case, $t\equiv\xi$, $p\equiv 2$, $q\equiv 1$, $\beta\equiv 1$, $\sigma\equiv\frac{r}{Z}+1$,

$$\alpha_p\equiv\left(\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z},\frac{Z+\rho}{Z}\right),$$

$$\rho_p\equiv\left(1+\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right),$$

and $Z \equiv -\frac{\varphi Z\xi}{\alpha \overline{A}\theta^{\alpha}}$ for [INT1], and $Z \equiv -\frac{\varphi Z\xi}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}$ for [INT2]. Then,

$$[INT1] = \frac{Z}{Z+\rho} {}_3F_2\left(a; b; -\frac{\varphi Z\xi}{\alpha \overline{A}\theta^{\alpha}}\right),$$

$$[INT1] = \frac{Z}{Z+\rho} {}_3F_2\left(a; b; -\frac{\varphi Z\xi}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right),$$

where

$$a = \left(1, \frac{Z+\rho}{Z}, \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right),$$

$$b = \left(1 + \frac{Z+\rho}{Z}, 1 + \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}\right).$$

By substituting this result into equation (b.2) we obtain [2] ■

c) Proof of Corollary 2

Under the conditions (A.22), (A.26), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right) > 0$, it is clear that Corollary 2 is verified if ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right) > 0$ (see equation (12)). Since we know that the denominator is positive, we have to verify that the numerator is positive too. To do this, we use the following property (see Abramowitz and Stegun (1970), 15.2.1, page 557):

$$\frac{d}{dz} {}_2F_1(a, b, c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; z) \qquad \text{(c.1).}$$

Since our a, b, c terms and ${}_2F_1(a, b, c; z)$ are positive, we conclude that ${}_2F_1(a+1, b+1, c+1; z)$ and the derivative are positive too. Then, ${}_2F_1(a, b, c; z)$ increases as z raises. Since $\exp(Zt)$ increases our z term, then

$${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}\exp(Zt)}\right) > {}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right) > 0$$

■

d) Monotonicity properties

Monotonicity of the ratio $X(t)$

From equation (B.1) (see Appendix B), we easily get that

$$\frac{dX(t)}{dt} = \varphi\left[\frac{Z}{\alpha}\frac{X(t)}{\varphi + \frac{\alpha \overline{A}\theta^{\alpha}}{Z}\exp(Zt)}\right]$$

Taking Corollary 7, we observe that the term between the square brackets is positive. Then, the sign of the derivative is determined by the sign of φ. If $\varphi < (>)0$, then $dX(t)/dt < (>)0$ for all t. Moreover, if $\varphi = 0$, then $X(t)$ is constant for all t. Since the sign of φ is determined by the ratio $S(0)/K(0)$, we can establish the following proposition:

Proposition d.1.

(1) *If $S(0)/K(0) > (<)(S(0)/K(0))^*$, then $X(t)$ decreases (increases) monotonically for all t.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$X(t) = \left(\frac{Z}{\alpha \overline{A}\theta^{\alpha}}\right)^{\frac{1}{\alpha}}$$

for all t.

Monotonicity of physical capital

Equation (12) provides the optimal solution path for $K(t)$. Let us study the detrended capital:

$$\tilde{K}(t) = K(0)(\varphi Z + \alpha \overline{A\theta}^{\alpha})^{\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \left(\frac{\varphi Z}{\exp(Zt) + \alpha \overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}. \tag{d.1}$$

From equation (d.1), following Abramowitz and Stegun (1970) (15.2.1, page 557), we get

$$\frac{d\tilde{K}(t)}{dt} = \tilde{K}(t) \frac{\varphi Z^2}{\alpha \exp(Zt)}$$

$$\cdot \left[\frac{Z+\rho}{2Z+\rho} \frac{1}{\overline{A\theta}^{\alpha}} \frac{{}_2F_1\left(a, b, c; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)}\right)} - \frac{1}{\alpha \overline{A\theta}^{\alpha} + \varphi Z \exp(Zt)}\right], \tag{d.2}$$

where

$$a = 2; b = 1 + \frac{Z+\rho}{Z}; c = b + 1.$$

Since ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)}\right) > 0$, applying the Euler integral representation as in Appendix A, we get that ${}_2F_1\left(a, b, c; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} \exp(Zt)}\right) > 0$. Therefore, following Corollary 7, all terms in equation (d.2) are positive. However, we can neither establish the sign of $d\tilde{K}(t)/dt$ nor conditions on the parameters to find out its sign. Nevertheless, we can conclude that $\tilde{K}(t)$ is not necessarily monotonic. Moreover, since $K(t) = \tilde{K}(t)\exp(\gamma_K t)$, where $\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho$, $K(t)$ is not necessarily monotonic either:[31]

Proposition d.2.

(1) *$\tilde{K}(t)$ is not necessarily monotonic.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $\tilde{K}(t) = K(0)$ for all t.*

(3) *$K(t)$ is not necessarily monotonic.*

(4) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $K(t)$ increases monotonically for all t.*

Monotonicity of investment

Since $\tilde{I}(t) = \overline{A}\tilde{K}(t) - \tilde{C}(t)$, then $d\tilde{I}/dt = \overline{A}d\tilde{K}(t)/dt - d\tilde{C}(t)/dt$. Hence, from the previous results, we conclude that the detrended investment is not necessarily monotonic.[32] Moreover, since $I(t) = \tilde{I}(t)\exp(\gamma_I t)$, the investment is not necessarily monotonic either:

Proposition d.3.

(1) *$\tilde{I}(t)$ is not necessarily monotonic.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$\tilde{I}(t) = \overline{A}K(0)\left(1 - \alpha\frac{Z+\rho}{Z}\right)$$

for all t.

(3) *$I(t)$ is not necessarily monotonic.*

(4) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $I(t)$ increases monotonically for all t.*

Monotonicity of output

Since the detrended output $\tilde{Y}(t)$ is equal to $\overline{A}\tilde{K}(t)$, then $d\tilde{Y}(t)/dt = \overline{A}d\tilde{K}(t)/dt$. As we showed before, we know that $\tilde{K}(t)$ is not necessarily monotonic. However, we can not establish a general condition for this non-monotonicity. Hence, we can only conclude that $\tilde{Y}$ is not necessarily monotonic.[33] Since $Y(t) = \tilde{Y}(t)\exp(\gamma_Y t)$, we conclude that $Y(t)$ is not necessarily monotonic either:

Proposition d.4.

(1) *$\tilde{Y}(t)$ is not necessarily monotonic.*

(2) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $\tilde{Y}(t) = \overline{A}K(0)$ for all t.*

(3) *$Y(t)$ is not necessarily monotonic.*

(4) *If $S(0)/K(0) = (S(0)/K(0))^*$, then $Y(t)$ increases monotonically for all t.*

Monotonicity of the stock of non-renewable energy resources

Equation (10) establishes that $\dot{S}(t) = -R(t)$. Then, $dS(t)/dt = -R(t)$. Therefore, since $R(t) > 0$ for all t, $dS(t)/dt < 0$ for all t:

Proposition d.5. *$S(t)$ decreases monotonically for all t.*

Notes

[1]They consider the mean of the period 1978-2005.

[2]Notice that, in general, the standard approach focuses on local dynamics, *i.e.*, around an equilibrium point (ex: steady state).

[3]See Stigliz (1974) for an economy with non-renewable resources and exponential growing labor force.

[4]If $A(t) = \overline{A}$ for all $t \geq 0$, it can be interpreted as a scale parameter.

[5]Further analysis could assume a CES function (in particular, Cobb-Douglas) where more inputs, such as energy (for final good production), or labour, would be included. In this case, there is an extra decision variable: the share of non-renewable energy resources in each sector. Moreover, due to the CES formulation, we would introduce an additional effect, namely, the substitutability among inputs in the final good production. However, the results concerning the energy-intensity hypothesis will remain qualitatively the same as in our simpler set-up. In any case, since the substitutability effect is not the central concern of this paper (see Krautkraemer (1998) for an accurate revision of the substitutability effect), we leave this analysis for future research.

[6]As we will see in Section 3, this paper considers technical progress ($A(t)$ and $\theta(t)$) as an exogenous variable.

[7]The standard approach considers that $Y(t) = f[K(t), R(t)]$, where $Y(t) = C(t) + I(t)$ and $\dot{K}(t) = I(t)$.

[8]If we consider the idea of a minimum amount of energy requiring to use a machine, the assumption of complementarity should be chosen (see for instance Pérez-Barahona and Zou (2006a,b)).

[9]Dasgupta and Heal (1974): "Extraction cost do not appear to introduce any great problem, provided that we assume any non-convexities". Indeed, we can easily introduce extraction cost $EC(t)$ by modifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t) + EC(R, S), \text{ where } \partial EC/\partial R > 0 \text{ and } \partial EC/\partial S \leq 0.$$

[10]Notice that there is no externality. Therefore, it is easy to prove that, under perfect competition, the decentralized solution is optimal.

[11]There are, indeed, two additional equations: a transversality condition (TC) for each state variable:

$$\lim_{t\to\infty}\psi(t)K(t)=0 \text{ and } \lim_{t\to\infty}\lambda(t)S(t)=0,$$

where $\psi(t)$ and $\lambda(t)$ are the corresponding Lagrangian multipliers. We use these two TC conditions to get the two constants involved in the calculation of the endogenous variables of the dynamical system. Actually, we can rewrite the previous system (6)-(10) as a system of seven equations (the five previous equations and the two TC) and seven unknowns (the former unknowns and the two constants related to each state variables).

[12]In Section 4, we will prove that there is asymptotic convergence to the BGP.

[13]Boucekkine and Ruiz-Tamarit (2008) also provide a quick overview of Gaussian hypergeometric functions.

[14]Recursivity is a necessary condition for the Gaussian hypergeometric representation.

[15]Since our problem is concave, the solution paths are unique. Thus, equation (A.25) has a unique solution for φ. A formal proof of uniqueness of the solution of equation (A.25) could be provided. However, this is beyond the scope of the present paper. The uniqueness can be tested by plotting equation (A.25) for given values of the parameters.

[16]Notice that $(Z+\Omega)\frac{Z+\rho}{Z}>0$ (see equation (A.21)). Moreover, the condition ${}_2F_1\left(-\frac{\varphi Z}{\alpha A\theta^{\alpha}}\right)>0$ is required to guarantee $C(0)>0$.

[17]The monotonicity properties of the other variables are available upon request (see Appendix E).

[18]Notice that Corollary 7 implies that both left and right hand side of (A.25) are positive.

[19]If $S(0)/K(0)=(S(0)/K(0))^*$ then detrended consumption keeps constant for all $t\geq 0$ at the corresponding BGP level, which equals $C(0)$.

[20]Unfortunately, we can not provide a general analytical conclusion for the effect of technical progress on t^*.

[21]Notice that 40.33% of the primary energy production (energy mix) of the EU in 2005 came from oil, while natural gas accounted for 21% (Eurostat (2008)). Therefore, oil and natural gas can be considered as the main primary energy resources in the EU (61.33% in 2005). We should also observe that the behaviour of supply and demand of natural gas in the EU is similar to oil (see IEA (2006) for further details).

[22]Notice that condition (A.22) implies $1>\alpha\frac{Z+\rho}{Z}$.

[23]Unfortunately, due to the complexity of equation (24), we can not provide further analytical results and interpretations on the monotonicity of $R(t)$.

[24]Notice that this is not a compulsory step. However, it provides simpler expressions by eliminating the long-run trends.

[25]This paper also provides a quick overview of Gaussian hypergeometric functions

[26]Notice that $K(t)=\tilde{K}(t)\exp\left[\left(\frac{1-\alpha}{\alpha}Z-\rho\right)t\right]$.

[27]This is because, on the one hand, non-renewable energy resources are necessary inputs to accumulate physical capital. On the other hand, energy-saving technical progress is incorporated into the economy through the production of new equipment. Therefore, since energy-saving technical progress grows with t, it is not optimal to deplete the stock of non-renewable energy resources in finite time.

[28]${}_3F_2(a;b;0)=1$ for a and b different than zero.

[29]Notice that, from Proposition 2 and the definition of Ω (see equation (A.15)), Ω should be less or equal to 0 to get BGP.

[30]If some particular parametrization generates a solution for the equation (A.25) out of the interval (A.26), one should study the so called Continuation formulas of the Gaussian hypergeometric representation (see Abramowitz and Stegun (1970)) for $S(t)$. However, this is beyond of the scope of the current paper.

[31]Notice that, as it is frequently assumed in models of energy economics (see for instance Dasgupta and Heal (1974), Sollow (1974a,b), Hartwick (1989), Pezzey and Withagen (1998) or Stokey (1998)), this paper does not consider physical capital depreciation. Therefore, $K(t)$ should be constant or increase monotonically. A sufficient condition for that is to consider a parametrization such that $\tilde{K}(t)$ increases monotonically. However, if we assume physical capital depreciation this condition is no longer required.

[32]We achieve the same conclusion by differentiating equation (13) with respect to t.

[33]As for investment, we get the same conclusion by differentiating equation (14) with respect to t.

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